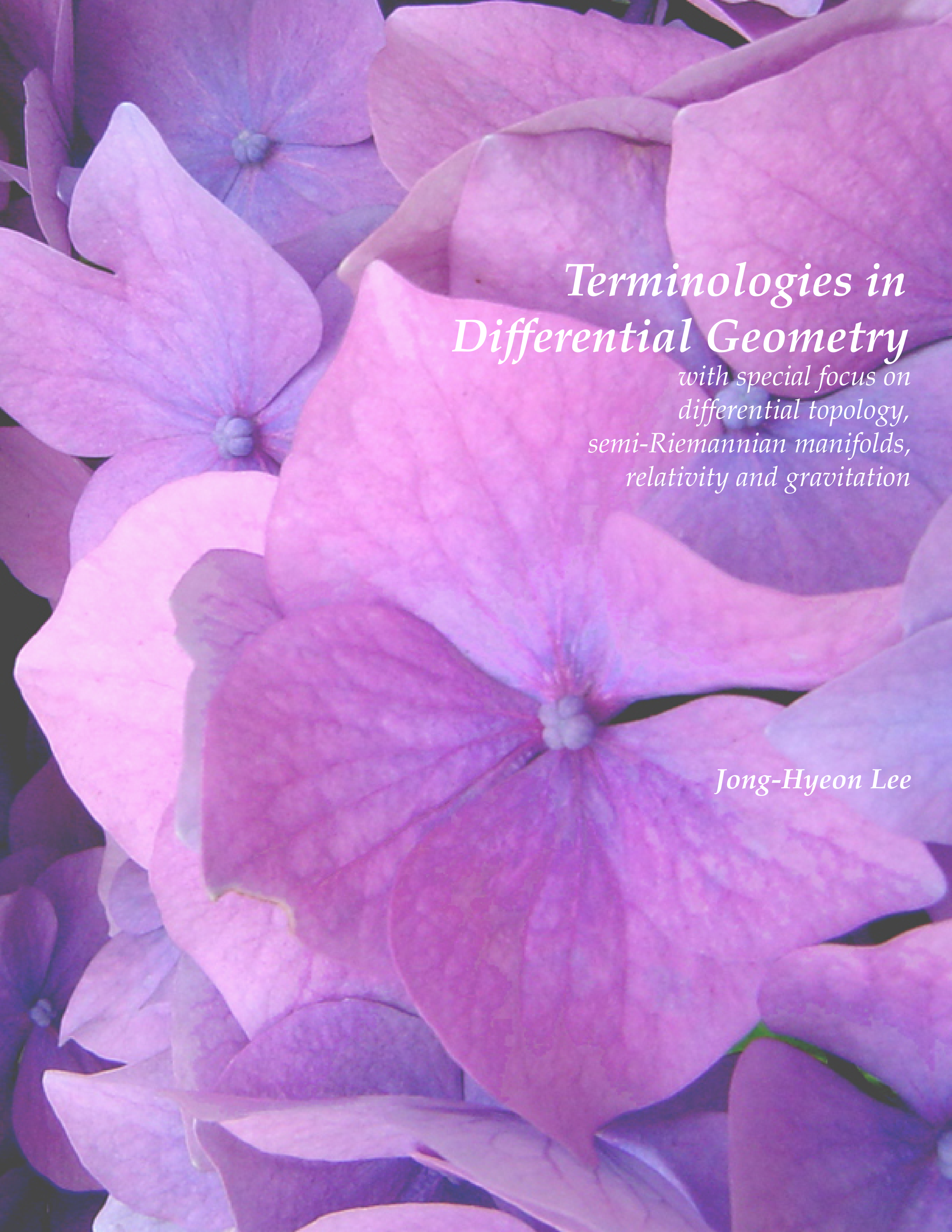

Terminologies in Differential Geometry

with special focus on differential topology, semi-Riemannian manifolds, relativity and gravitation

Jong-Hyeon Lee

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To my parents and family

Preface

There are many ways to represent a phenomenon in various ways. Authors of each book would like to find the best way to represent their ideas to convey them as clear as possible. As an unexpected result, we often face different types of notations and definitions describing the same concept. It makes us difficult to grasp the concept of the terms.

Differential geometry is not an exception, especially, semi-Riemannian geometry due to its inter-disciplinary nature. Terms used in physics are not defined in the same way as in mathematics, *vice versa*. I believe that it would be helpful to have a collection of terminologies described in a uniform idea and notions. I collected and re-wrote terminologies used in differential geometry, especially in differential topology and semi-Riemannian manifolds. Some terms used in relativity and gravitation were added to this book and were represented in mathematical viewpoints rather than in physical viewpoints, since physical representations often become a barrier for mathematicians to understand the underlying mathematical meaning.

A number of papers and books on semi-Riemannian geometry, differential topology, relativity and gravitation were reviewed for this book. Among them, the notations and definitions of Barret O'Neill's seminal book *Semi-Riemannian Geometry* (Academic Press, 1983) became a basis of representing terminologies in this book. For terms not covered by the O'Neill's book, I tried to describe them in the way represented in his book as consistent as possible.

I wish this book would be any help for readers who study differential geometry and its applications.

J.-H. Lee

Taejon, Korea
March, 1993

A

abstract simplicial complex An *abstract simplicial complex* is a collection $\mathcal{S}$ of finite nonempty sets such that if A is an element of $\mathcal{S}$, so is every nonempty subset of A. The element A of $\mathcal{S}$ is called a *simplex of* $\mathcal{S}$; its dimension is one less than the number of its element. Each nonempty subset of A is called a *face of* A. The faces of A different from A itself are called the *proper faces of* A and their union is called the *boundary of* A and denoted by bd$A$. The *vertex set* V *of* $\mathcal{S}$ is the union of one-point elements of $\mathcal{S}$. A subcollection of $\mathcal{S}$ that is itself a complex is called a *subcomplex of* $\mathcal{S}$.

acausal A subset A of M is *acusal*, provided that the relation $p < q$ never holds for $p, q \in A$, that is, provided that no causal curve meets A more than once.
facta. This is stronger requirement than achronality.

achronal A subset A of M is *achronal*, provided that the relation $p \ll q$ never holds for $p, q \in A$, that is, provided that no timelike curve meets A more than once.
facta. In $\mathbf{R}^n_1$, a hyperplane t constant is achronal.

action *refer to* energy

action of Lie group A *(left) action of a Lie group* G *on a manifold* M is a smooth map $G \times M \longrightarrow M$, denoted by $(g, p) \rightarrow gp$, such that

i. $(gh)p = g(hp)$ for all $g, h \in G$ and $p \in M$.
ii. $ep = p$ for all $p \in M$, where e is the identity element of G.

Here G is also called a *Lie transformation group on* M. An action $G \times M \longrightarrow M$ is *transitive* provided that for each $p, q \in M$, there is a $g \in G$ such that $gp = q$. If $G \times M \longrightarrow M$ is an action and p a point in M, then $H = \{g \in G : gp = p\}$ is a closed subgroup of G called the *isotropy subgroup at* p.

adapted normal neighborhood A convex normal neighborhood U of M with compact closure $\overline{U}$ is called an *adapted normal neighborhood* if $\overline{U}$ is covered by adapted coordinates $(x_i)_{1 \leq i \leq n}$ which are adapted at some point of U such that the followings hold:

i. At every point of U, the components $\mathbf{g}_{ij}$ of the metric tensor $\mathbf{g}$ expressed in the given coordinates $(x_i)_{1 \leq i \leq n}$ differ from the matrix diag$\{-1, +1, \ldots, +1\}$ by at most 1/2.
ii. The metric $\mathbf{g}$ satisfies $\mathbf{g} <_U \mathbf{g}_0$, where $\mathbf{g}_0$ is Minkowski metric $\mathrm{d}s^2 = -2\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2$ for U.

$\mathsf{Ad}(H)$-invariant Let H be a subgroup of G. An object defined on the Lie algebra $\mathfrak{g}$ of G is *$\mathsf{Ad}(H)$-invariant* if it is preserved by $\mathsf{Ad}_h : \mathfrak{g} \longrightarrow \mathfrak{g}$ for all $h \in H$.

adjoint mapping Ad_a *refer to* adjoint representation of Lie group

adjoint representation of a Lie group Let G be a Lie group and $a \in G$. Let $C_a : G \longrightarrow G$ be the function sending each g to aga^{-1}. Then C_a is an automorphism. The differential of C_a is denoted by Ad_a and is called an *adjoint mapping*. Then $C_{ab} = C_a \circ C_b$. The homomorphism $a \mapsto \mathsf{Ad}_a$ is called the *adjoint representation of G*.

adjunction space Let X and Y be disjoint topological spaces and let A be a closed subset of X. If $f : A \longrightarrow Y$ is a continuous map, let's consider a quotient space like below:
Topologize $X \cup Y$ as the topological sum. Form a quotient space by identifying each set

$$\{y\} \cup f^{-1}(y)$$

for $y \in Y$, to a point. That is, partition $X \cup Y$ into these sets, along with the one-point sets $\{x\}$, for $x \in X - A$. This quotient space is denoted by $X \cup_f Y$ and is called the *adjoint space determined by f*.

admissible class of spaces Let $\mathcal{A}$ be a class of pairs (X, A) of topological spaces such that

i. If (X, A) belongs to $\mathcal{A}$, so do (X, X), $(X, \emptyset)$, (A, A) and $(A, \emptyset)$.
ii. If (X, A) belongs to $\mathcal{A}$, so does $(X \times I, A \times I)$.
iii. There is an one-point space P such that $(P, \emptyset)$ is in $\mathcal{A}$.

Then $\mathcal{A}$ is called an *admissible class of spaces for a homology theory*.

Ado's theorem There is an one-to-one correspondence between isomorphism classes of Lie algebras and isomorphism classes of simply connected Lie groups.

affine parameter Any parameter which makes a smooth curve a geodesic is called an *affine parameter*.

affine transform An *affine transformation T of* $\mathbf{R}^n$ is a map that is a composition of translations (i.e., maps of the form $T(x) = x + p$ for fixed p) and nonsingular linear transformations.

Alexander duality Let n be fixed. There is a function assigning to each proper nonempty polyhedron A in S^n, an isomorphism

$$\alpha_A : \widetilde{H}^k(A) \longrightarrow \widetilde{H}_{n-k-1}(S^n - A).$$

This assignment is natural with respect to inclusions.

Alexander-Pontrjagin duality Let A be a proper nonempty closed subset of S^n. Then

$$\check{H}^k(A) \simeq \widetilde{H}_{n-k-1}(S^n - A),$$

where $\check{H}^k(A)$ is a Čech cohomology group and $\widetilde{H}_{n-k-1}(S^n - A)$ is a singular homology group.

Alexander-Spanier cohomology Let M be a paracompact Housdorff space and K a fixed principal ideal domain. Now define the *classical Alexander-Spanier cohomology modules of M with coefficients in a K-module G*. Let $A^p(U,G)$ denote the K-module of functions $U^{p+1} \longrightarrow G$ and let

$$A_0^p(M,G) = \{f \in A^p(M,G) : \rho_{m,M}(f) = 0 \text{ for all } m \in M\}.$$

Define a homomorphism

$$d : A^p(M,G) \longrightarrow A^{p+1}(M,G)$$

by

$$df(m_0, \ldots, m_{p+1}) = \sum_{i=0}^{p+1} (-1)^i f(m_0, \ldots, \widehat{m_i}, \ldots, m_{p+1})$$

for each $f \in A^p(M,G)$ and $(m_0, \ldots, m_{p+1}) \in M^{p+2}$, where $\widehat{\ }$ over an entry means that this entry is to be omitted. Then this homomorphism restricted to $A_0^p(M,G)$ has range in $A_0^{p+1}(M,G)$ and thus yields homomorphisms on quotients

$$A^p(M,G)/A_0^p(M,G) \longrightarrow A^{p+1}(M,G)/A_0^{p+1}(M,G).$$

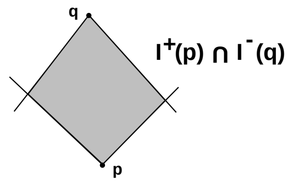
Above sequence of modules and homomorphisms for $p \geq 0$ form a cochain complex which we shall denote by $A^\star(M,G)/A_0^\star(M,G)$ and in which the modules of q cochaons for $q < 0$ are as usual, all assumed to be zero. The *classical cohomology modules for M with coefficient in the K-module G* are given by

$$H_{A-S}^q(M;G) = H^q(A^\star(M,G)/A_0^\star(M,G)).$$

facta. When $\mathcal{G} = M \times G$, we have canonical isomorphisms

$$H_{A-S}^q(M;G) \simeq H^q(M, \mathcal{G}).$$

Alexandrov topology The *Alexandrov topology on an arbitrary spacetime* $(M, \mathbf{g})$ is the topology given M by taking as a basis all sets of the type $I^+(p) \cap I^-(q)$, for $p, q \in M$.
rel. strong causality condition



alternating multilinear map A multilinear map

$$h : \underbrace{V \times \cdots \times V}_{r\text{-times}} \longrightarrow W$$

is called *alternating* if

$$h(v_{\pi(1)}, \ldots, v_{\pi(r)}) = (\text{sgn } \pi) h(v_1, \ldots, v_r)$$

for $v_1, \ldots, v_n \in V$ and all permutations π in the permutation group S_r on r letters. The *sign of the permutations* π is

$$\text{sgn } \pi = \begin{cases} +1 & \text{if } \pi \text{ is even,} \\ -1 & \text{if } \pi \text{ is odd.} \end{cases}$$

analytic extension of a manifold *refer to* extension of a manifold

angular momentum Let α be a particle with mass $m \ll \mathsf{M}$ in $\mathbf{R}^3$, where M is the mass of the sun. Then the vector field $\widetilde{L} = \alpha \times \alpha'$ is the *angular momentum vector of* α *per unit mass*.
facta. If $\widetilde{L} = 0$, then α lies in a line through the origin, hence we can assume that α lies in the xy-plane of $\mathbf{R}^3$. Then the *angular momentum of* α *per unit mass* is the number L such that $\widetilde{L} = L\partial_z$.

anti-derivation in exterior algebra *refer to* exterior algebra endomorphisms

anti-isometry *refer to* homothety

antipodal map The *antipodal map* $a : S^n \longrightarrow S^n$ is the map defined by $a(x) = -x$ for all x. And the *degree of antipodal map* is $(-1)^{n+1}$.

antipode-preserving map A map $f : S^n \longrightarrow S^n$ is said to be *antipodal-preserving* if $f(-x) = -f(x)$ for all x in S^n.
cf. antipodal map

arclength Let $\alpha : [a, b] \longrightarrow M$ be a piecewise smooth curve segment in a semi-Riemannian manifold M. The *arclength of* α is

$$L(\alpha) = \int_b^a |\alpha'(s)| \, \mathrm{d}s.$$

asymptotically stable point Let p be a critical point of vector field X. Then p is *asymptotically stable* if there is a neighborhood V of p such that if $q \in V$, then q is complete to positive direction, $\psi_t(V) \subset \psi_s(V)$ if $t > s$ and

$$\lim_{t \to +\infty} \psi_t(V) = \{p\}.$$

That is, for any neighborhood U of p, there is a number T such that $\psi_t(V) \subset U$ if $t \geq T$.
rel. stable point

atlas An *atlas* $\mathcal{A}$ of dimension *n* on a space $\mathcal{S}$ is a collection of *n* -dimensional coordinate systems in $\mathcal{S}$ such that

i. each point of $\mathcal{S}$ is contained in the domain of some coordinate system in $\mathcal{V}$,

ii. any two coordinate systems in $\mathcal{A}$ overlap smoothly.

augmentation map Let K be a complex and $C_p(K)$ a group of p-chains. Let $\epsilon : C_0(K) \longrightarrow \mathrm{Z}$ be the surjective homomorphism defined by $\epsilon(v) = 1$ for each vertex v of K. Then if c is a 0-chain, $\epsilon(c)$ equals the sum of the values of c on the vertices of K. The map ϵ is called an *augmentation map for* $C_0(K)$. The *reduced homology group* $\widetilde{H}_0(K)$ *of* K *in dimension zero* is deined by

$$\widetilde{H}_0(K) = \ker\epsilon / \mathrm{im}\partial_1.$$

If $p > 0$, we let $\widetilde{H}_p(K)$ denote the usual homology group $H_p(K)$.

automorphism of a Lie group An *automorphism of a Lie group* G is a map $\phi : G \longrightarrow G$ that is both a diffeomorphism and a group isomorphism.

Axiom of compact support *refer to* homology theory

B

baby ham sandwich theorem Let A_1 and A_2 be two bounded measurable subsets of $\mathbf{R}^2$. There is a line in $\mathbf{R}^2$ that bisects both A_1 and A_2.

backward Schwarz inequality Let v and w be timelike vectors in a Lorentz vector space . Then

$$|\langle v, w\rangle| \geq |v|\,|w|\,,$$

with equality if and only if v and w are collinear. This inequality is called *backward Schwarz inequality*. Sometimes this inequality is called the *reverse Schwarz inequality*.

backward triangle inequality Let v and w be timelike vectors in a Lorentz vector space . If v and w are in the same timecone, then

$$|v| + |w| \leq |v + w|$$

with equality if and only if v and w are collinear. This inequality is called *backward triangle inequality*. In terms of Lorentzian distance function d, this inequality would be expressed by

$$d(p, q) \geq d(p, r) + d(r, q)$$

whenever $p \leq r \leq q$. Sometimes this inequality is called the *reverse triangle inequality (RTI)*.

b.a.complete A spacetime $(M, \mathbf{g})$ is called *b.a.complete* if all future [past] pointing, future [past] inextendible unit speed C^2 timelike curves with bounded acceleration have infinite length. If there exists a future [past] pointing, future [past] inextendible unit speed C^2 timelike curve with bounded acceleration but finite length, then $(M, \mathbf{g})$ is called *b.a.incomplete*.

b.a.incomplete *refer to* b.a.complete

Baire category theorem Every complete pseudometric space is a Baire space.

Baire space Let X be a topological space and $A \subset X$. Then A is called *residual* if A is the intersection of a countable family of open dense subsets of X. A space X is called a *Baire space* if every residual set is dense.

barycentric coordinate If x is a point of the polyhedron $|K|$, then x is interior to precisely one simplex of K, whose vertices are (say) $a_0, \ldots, a_n$. Then

$$x = \sum_{i=0}^{n} t_i a_i,$$

where $t_i > 0$ for each i and $\sum t_i = 1$. If v is an arbitrary vertex of K, we define the *barycentric coordinate* $t_v(x)$ *of* x *with respect to* v by setting $t_v(x) = 0$ if v is not one of the vertices a_i and $t_v(x) = t_i$ if $v = a_i$.
cf. n-simplex

base curve *refer to* variation

base manifold *refer to* vector bundle

base of the cone *refer to* cone on a complex

basis Let S be a topological space. Then a *basis* for the topology is a collection $\mathcal{B}$ of open sets such that every open sets of S is a union of elements of $\mathcal{B}$.

b-boundary extension Let $\gamma : [0, a) \longrightarrow M$ be a b-incomplete curve which is not extendible to $t = a$ in M. A *local b-boundary extension about* γ is an open neighborhood $U \subset M$ of γ and an extension $\widetilde{U}$ of U such that the image of γ in $\widetilde{U}$ is inextendible continuously beyond $t = a$.

b-complete The spacetime $(M, \mathbf{g})$ is called *b-complete* if every C^1 curve of finite arclength as measured by a generalized affine parameter has an endpoint in M.

Betti number *refer to* Fundamental theorem of finitely generated abelian groups

(first) Bianchi identity *refer to* Lie bracket

(second) Bianchi identity If $x, y, z \in T_pM$, the equation

$$(D_z\mathcal{R})(x, y) + (D_x\mathcal{R})(y, z) + (D_y\mathcal{R})(z, x) = 0$$

holds and is called the *second Bianchi identity*.

big bang An initial singularity of Robertson-Walker spacetime $M(k, f)$ at $t_\star$ is a *big bang* provided $f \rightarrow 0$ and $f' \rightarrow \infty$ as $t \rightarrow t_\star$. Similarly, a final singularity is a *big crunch* if $f \rightarrow 0$ and $f' \rightarrow -\infty$ as $t \rightarrow t^\star$, where $I = (t_\star, t^\star)$.

big crunch *refer to* big bang

bi-invariant *refer to* left-invariant

bilinear form A *bilinear for on a vector space* V is an $\mathbf{R}$-bilinear function $b : V \times V \longrightarrow \mathbf{R}$.
facta. A bilinear for is a (0,2) tensor field .

Bochner's theorem On a compact Riemannian manifold with $\mathrm{Ric} < 0$, every Killing vector field is identically zero.

Bolzano-Weierstrass theorem If S is a first countable space and is compact, then every sequence has a convergent subsequence.

Borsuk-Ulam theorem If $h : S^n \longrightarrow \mathbf{R}^n$ is a continuous map, then $h(x) = h(-x)$ for at least one $x \in S^n$.

boundary in homology *refer to* chain complex

boundary of a manifold Let M be an n-manifold with boundary. If the point p in M maps to a point of bdH^n under one coordinate system about p, then it maps to a point of bdH^n under such coordinate system. Such a point is called the *boundary of* M and is denoted by bd$M$ or ∂M. The space "$M -$ bd$M$" is called the *interior of* M and denoted by int M.
rel. manifold with boundary

boundary of a set Let S be a topological space and $A \subset S$. The *boundary of* A, denoted bd$A$, is defined by

$$\text{bd}A = \overline{A} \cap \overline{\mathcal{C}A}.$$

facta. bd$A$ is closed and bd$A =$ bd$\mathcal{C}A$.

boundary of a simplex *refer to* abstract simplicial complex

boundary operator *refer to* chain complex

bounded acceleration A C^2 timelike curve $\gamma : I \longrightarrow M$ with $\mathbf{g}(\gamma'(t), \gamma'(t)) = -1$ for all $t \in I$ is said to have *bounded acceleration* if there is a constant $B > 0$ such that $|\mathbf{g}(D_{\gamma'}\gamma'(t), D_{\gamma'}\gamma'(t))| \leq B$ for all $t \in I$. Here D is the unique torsion-free connection for M defined by the metric g.

bounded chain *refer to* homologous chains

bracket operation *refer to* Lie algebra

Brouwer fixed-point theorem Every continuous map $\phi : B^n \longrightarrow B^n$ has a fixed-point.

bump function Given any neighborhood U of a point p in M, there is a function $f \in \mathcal{F}(M)$, called *bump function at* p such that

i. $0 \leq f \leq 1$ on M,

ii. $f = 1$ on some neighborhood of p,

iii. $\mathsf{supp} f \subset U$.

rel. partition of unity

bundle chart *refer to* vector bundle

Busemann function Let $\gamma : I \longrightarrow M$ be a piecewise smooth timelike curve. The *(future/past) Busemann functions* $b^{\pm}_{\gamma} : J^{\mp}(\gamma) \longrightarrow \mathbf{R} \cup \{\mp\infty\}$ of γ are defined by

$$b^{+}_{\gamma}(p) = \inf_{t \in I} b^{+}_{\gamma,t}(p), \textit{ (future Busemann function)}$$

and

$$b^{-}_{\gamma}(q) = \sup_{t \in I} b^{-}_{\gamma,t}(q). \textit{ (past Busemann function)}$$

rel. pre-Busemann function

C

canonical projection *refer to* equivalence relation

Cartan lemma Let $p \leq d$ and let $\omega_1, \ldots, \omega_n$ be one-forms on d-manifold M which are linearly independent pointwise. Let $\theta_1, \ldots, \theta_p$ be one-forms on M such that

$$\sum_{i=1}^{p} \theta_i \wedge \omega_i = 0.$$

Then there exist smooth functions A_{ij} on M with $A_{ij} = A_{ji}$ such that

$$\theta_i = \sum_{j=1}^{p} A_{ij}\omega_j$$

for $i = 1, \ldots, p$.

category A *category* C consists of three things:

i. A class of *objects* X.
ii. For every ordered pair (X, Y) of objects, a set $\hom(X, Y)$ of *morphisms* f.
iii. A function, called *composition of morphisms*,

$$\hom(X, Y) \times \hom(Y, Z) \longrightarrow \hom(X, Z),$$

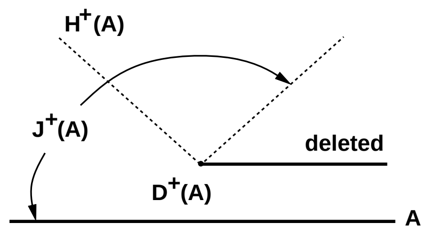
which is defined for every triple (X, Y, Z) of objects.

Cauchy development If A is an achronal subset of M, the *future Cauchy development of* A is the set $D^+(A)$ of all points p of M such that every past-inextendible causal curve through p meets A.
With the *past Cauchy development* $D^-(A)$ defined dually, $D(A) = D^-(A) \cup D^+(A)$ is the *Cauchy development of* A.
facta. 1. $A \subset D(A)$.
2. An achronal set A in a semi-Riemannian manifold M is a Cauchy hypersurface if and only if $D(A) = M$.

Cauchy horizon If A is an achronal set, its *future Cauchy horizon* $H^+(A)$ is

$$\overline{D^+}(A) - I^-(D^+(A)) = \left\{ p \in \overline{D^+}(A) : I^+(p) \text{ does not meet } D^+(A) \right\}.$$

With the *past Cauchy horizon* $H^-(A)$ defined dually, the *Cauchy horizon* of A is $H(A) = H^-(A) \cup H^+(A)$.
facta. 1. If $H^+(A)$ is nonempty, the entire future of A cannot be predicted from A.
2. $H^+(A)$ separates $D^+(A)$ from the rest of $J^+(A)$.
3. For a topological hypersurface A, A is a Cauchy hypersurface if and only if $H(A)$ is empty.



Cauchy hypersurface A *Cauchy hypersurface in* M is a subset S that is met exactly once by every inextendible timelike curve in M.
facta. 1. S is closed achronal topological hypersurface and is met by every inextendible causal curve .
2. Any two Cauchy hypersurfaces in M are homeomorphic .
3. The strong causality condition holds on any simply connected Lorentz surface.

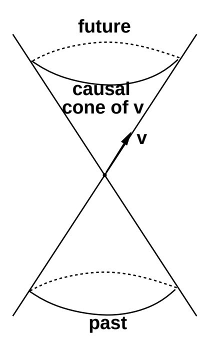
Cauchy-Riemann operator The *Cauchy-Riemann operator* is

$$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

facta. Cauchy-Riemann operator is elliptic.

causal boundary The *causal boundary of a spacetime* (M, g) will be denoted by $\mathrm{bd}_c M$. It is formed by indecomposable past [future] sets which do not correspond to the past [future] of any point of M.
facta. It is invariant under conformal changes.

causal cone For a timelike vector v the set $C(v)$ of all causal vectors w such that $\langle v, w\rangle < 0$ is the *causal cone containing* v.



causal curve In a Lorentz manifold a *causal curve* is one whose velocity vectors are all nonspacelike.

causal cut locus *refer to* future causal cut locus

causal future *refer to* chronological future

causal past *refer to* chronological future

causal geodesically complete A semi-Riemannian manifold M is said to be *causal geodesically complete* if all causal inextendible geodesics are complete. Sometimes a causal geodesically completeness is called a *nonspacelike geodesically completeness*.

causal geodesically incomplete A semi-Riemannian manifold M is said to be *causal geodesically incomplete* if some causal geodesic is incomplete. Sometimes a causal geodesically incompleteness is called a *nonspacelike geodesically incompleteness*.

causality condition The manifold M satisfies the *causality condition* provided there are no closed causal curves in M.

causality relation The *causality relations on* M are defined as follows. If $p, q \in M$, then

i. $p \ll q$ means there is a future pointing *timelike* curve in M from p to q.

ii. $p < q$ means there is a future pointing *causal* curve in M from p to q.

facta. As usual, $p \leq q$ means that either $p < q$ or $p = q$.

causally convex An open set U in a spacetime is called *causally convex* if no causal curve intersects U in a disconnected set.

causally disconnected A spacetime $(M, \mathbf{g})$ is called *causally disconnected by a compact set* K if there are two infinite sequences $\{p_n\}$ and $\{q_n\}$, both diverging to infinity, such that $p_n \leq q_n$, $p_n \neq q_n$ and all future pointing causal curves from p_n to q_n meet K for each n.
A spacetime $(M, \mathbf{g})$ admitting such a compact K causally disconnecting two divergent sequences is called *causally disconnected*.

causally simple A distinguishing spacetime $(M, \mathbf{g})$ is *causally simple* if $J^+(p)$ and $J^-(p)$ are closed subsets of M for all $p \in M$.

causal vector In a Lorentz vector space a vector that is nonspacelike is called *causal*.

centered coordinate system Let φ be a coordinate system of an open set U. If $m \in U$ and $\varphi(m) = 0$, then the coordinate system is said to be *centered at* m.

chain A *p-chain C in M with K-coefficients* is a finite linear combination $c = \sum_i a_i\sigma_i$ of p-simplices σ_i in M for $a_i \in K$.

chainable matched covering A matched covering $(U^\star, \sim)$ of a manifold M is *chainable over a curve* $\sigma : [0,1] \longrightarrow M$ provided that given any $a \in A$ such that $\sigma(0) \in U_a$, there exist numbers $0 = t_0 < t_1 < \cdots > t_k = 1$ and indices $a = a_1 \sim a_2 \sim \cdots \sim a_k$ such that

$$\sigma([t_{i-1}, t_i]) \subset U_{a_i} \text{ for } \ 1 \le i \le k.$$

chain complex A *chain complex* $\{\mathcal{C}, \partial\}$ is a sequence

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots$$

of abelian groups C_i and homomorphisms ∂_i, indexed with the integers, such that $\partial_p \circ \partial_{p+1} = 0$ for all p. The *p-th homology group of $\mathcal{C}$* is defined by

$$H_p(\mathcal{C}) = \ker\partial_p / \mathrm{im}\partial_{p+1}.$$

The map ∂_p is called *p-th boundary operator*. Elements of $\ker\partial_p$ are called the *differentiable p-cycles* and elements of $\mathrm{im}\partial_{p+1}$ are called *differentiable p-boundaries*.

chain equivalence A chain map $\phi : \mathcal{C} \longrightarrow \mathcal{C}'$ is called a *chain equivalence* if there is a chain map $\phi' : \mathcal{C}' \longrightarrow \mathcal{C}$ such that $\phi' \circ \phi$ and $\phi \circ \phi'$ are chain homotopic to the identity maps of $\mathcal{C}$ and $\mathcal{C}'$, respectively. The map ϕ' is called a *chain homotopy inverse of ϕ*.

chain homotopy If $\phi, \psi : \mathcal{C} \longrightarrow \mathcal{C}'$ are chain maps, then a *chain homotopy of ϕ to ψ* is a family of homomorphisms

$$D_p : C_p \longrightarrow C'_{p+1}$$

such that

$$\partial'_{p+1} D_p + D_{p+1}\partial_p = \psi_p - \phi_p$$

for all p.

chain homotopy inverse *refer to* chain equivalence

chain map Let $\mathcal{S} = \{C_p, \partial_p\}$ and $\mathcal{C}' = \{C'_p, \partial'\}$ be chain complexes. A *chain map* $\phi : \mathcal{C} \longrightarrow \mathcal{C}'$ is a family of homomorphisms $\phi_p : C_p \longrightarrow C'_p$ such that

$$\partial'_p \circ \phi_p = \phi_{p-1} \circ \partial_p \ \text{ for all } p.$$

chain-rule If $\phi : M \longrightarrow N$ and $\psi : N \longrightarrow P$ are smooth mappings, then for each $p \in M$,

$$\mathrm{d}(\psi \circ \phi)_p = \mathrm{d}\psi_{\phi(p)} \circ \mathrm{d}\phi_p.$$

This expression is called the *chain-rule*.
facta. The classical chain-rule formula is expressed by the Jacobian matrix for a composite mapping.

characteristic bundle Let ω be an exterior two-form on M. Then

$$R_\omega = \Big\{ v \in TM : v^\flat = 0 \Big\}$$

is called the *characteristic bundle of* ω. Here $v^\flat$ is the one-form defined by $v^\flat(\omega) = \omega(v, w)$. A *characteristic vector field* is a vector field X such that $i_X\omega = 0$; that is, $X(\omega) \in R_\omega$ for all points in M.

characteristic vf *refer to* characteristic bundle

chart *refer to* coordinate system

Christoffel symbol Let $x^1, \ldots, x^n$ be a coordinate system on a neighborhood U in a semi-Riemannian manifold M. The *Christoffel symbols* for this coordinate system are the real-valued functions Γ^k_{ij} on U such that

$$D_{\partial_i}(\partial_j) = \sum_k \Gamma^k_{ij}\partial_k \ \ (1 \le i, j \le n).$$

chronological future For a subset A of M, the subset

$$I^+(A) = \{q \in M : \text{there is a} \ \ p \in A \text{ with } \ \ p \ll q\}$$

is called the *chronological future* of A, and

$$J^+(A) = \{q \in M : \text{there is a} \ \ p \in A \text{ with } \ \ p \le q\}$$

is called the *causal future* of A. Dually, *chronological past* $I^-(A)$ of A and *causal past* $J^-(A)$ of A would be defined.

chronological past *refer to* chronological future

chronology condition If M contains no closed timelike curves , we say that the *chronology condition* holds on M.

class C^k Let U be an open set in $\mathbf{R}^n$ and let $f : U \longrightarrow \mathbf{R}$. We say that f is *differentiable of class* C^k *on* U for a nonnegative integer k, if the partial derivatives $\partial^\alpha f/\partial r^\alpha$ exist and are continuous on U for $[\alpha] \le k$, where $[\alpha] = \sum_i \alpha_i$ and α is a n-tuple of nonnegative integers. In particular, f is C^0 if f is continuous. For $f : U \longrightarrow \mathbf{R}^m$, f is differentiable of class C^k if each of the component functions $f_i = u^i \circ f$ is C^k, where u^i is a natural coordinate function of $\mathbf{R}^m$. We also say that f is C^∞ if it is C^k for all $k \ge 0$.

Clifton-Pohl torus Let M be $\mathbf{R}^2 - 0$ with $\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v/(u^2 + v^2)$. The scalar multiplication by any $c \ne 0$ is an isometry of M. Take $\nu(u, v) = (2u, 2v)$. The group $\Gamma = \{\mu^n\}$ generated by μ is properly discontinuous; thus $T = M/\Gamma$ is a Lorentz surface. Topologically T is the closed annulus $1 \le r \le 2$ with boundary points identified underμ. Thus T is a torus. T is called the *Clifton-Pohl torus*.
facta. T is compact but not complete.

clean intersection Let L and M be submanifolds of P. L and M have *clean intersection* if $L \cap M$ is a submanifold and $TL \cap TM = T(L \cap M)$.

closed form Let A be open in M. A k-form ω on A with $k \geq 0$ is said to be *closed* if $\mathrm{d}\omega = 0$.
cf. exact form

closed set *refer to* topological space

closed star of a simplex *refer to* star of a simplex

closed star of a vertex *refer to* star of a vertex

closed subgroup A *closed subgroup H of a Lie group G* is an abstract subgroup that is a closed subset of G.

closure of a set Let S be a topological space and $A \subset S$. Then the *closure of A*, denoted $\overline{A}$, is the intersection of all closed sets containing A.
cf. interior of a set
facta. If A is closed, $\overline{A} = A$.

coboundary operator *refer to* cochain complex

cochain complex Let $\mathcal{C} = \{C_p, \partial\}$ be a chain complex and let G be an abelian group. The *p-dimensional cochain group of $\mathcal{C}$ with coefficients in G* is

$$C^p(\mathcal{C}; G) = \mathsf{Hom}\ (C_p, G).$$

The *coboundary operator* δ is the dual of the boundary operator of chain complex. The family of groups and homomorphisms $\{C^p(\mathcal{C}; G), \delta\}$ is called the *cochain complex of $\mathcal{C}$ with coefficients in G*.
As usual, the kernel of the homomorphism

$$\delta : C^p(\mathcal{C}; G) \longrightarrow C^{p+1}(\mathcal{C}; G)$$

is denoted by $Z^p(\mathcal{C}; G)$ and its image is denoted by $B^{p+1}(\mathcal{C}; G)$. The *cohomology group of $\mathcal{C}$* in dimension p with coefficients in G is defined by

$$H^p(\mathcal{C}; G) = Z^p(\mathcal{C}; G)/B^p(\mathcal{C}; G).$$

If $\{\mathcal{C}, \epsilon\}$ is an augmented chain complex, then one has a corresponding chain complex

$$\cdots \longleftarrow C'(\mathcal{C}; G) \xleftarrow{\delta_1} C^0(\mathcal{C}; G) \xleftarrow{\tilde{\epsilon}} \mathsf{Hom}\ (\mathsf{Z}, G),$$

where $\tilde{\epsilon}$ is one-to-one. We define the *reduced cohomology groups of $\mathcal{C}$* by setting $\widetilde{H}^q(\mathcal{C}; G) = H^q(\mathcal{C}; G)$ if $q > 0$ and

$$\widetilde{H}^0(\mathcal{C}; G) = \ker\delta_1/\mathrm{im}\tilde{\epsilon}.$$

In general, we have the equation

$$H^0(\mathcal{C};G) \simeq \widetilde{H}^0(\mathcal{C};G) \oplus G.$$

cf. chain complex

cochain group *refer to* cochain complex

cochain map Let $\mathcal{C} = \{C_p, \partial\}$ and $\mathcal{C}' = \{C'_p, \partial'\}$ be chain complexes. If $\phi : \mathcal{C} \longrightarrow \mathcal{C}'$ is a chain map, then $\partial' \circ \phi = \phi \circ \partial$. Then the dual homomorphism

$$C^p(\mathcal{C};G) \xleftarrow{\tilde{\phi}} C^p(\mathcal{C}';G)$$

commutes with δ. Such a homomorphism is called a *cochain map*.
cf. chain map

Codazzi equation Let M be a semi-Riemannian submanifold of $\overline{M}$. For $V, W, X \in \mathfrak{X}(M)$, the following equation, called *Codazzi equation* , holds.

$$\text{nor}\, \overline{\mathcal{R}}_{VW}X = -(\bigtriangledown_V II)(W, X) + (\bigtriangledown_W II)(V, X),$$

where $(\bigtriangledown_V II)(X, Y) = D^{\perp}_V(II(X, Y)) - II(D_V X, Y) - II(X, D_V Y)$.
rel. normal connection

codifferential operator Using the Hodge star operator $\star$, let $(\alpha, \beta) = \int_M \alpha \wedge \star\beta$ which gives an L^2 inner product on the sections of exterior algebra of degree k, $\Lambda^k(M)$. The *codifferential operator*δ is defined by

$$\delta = (-1)^{n(k+1)+1} \star \mathrm{d} \star .$$

facta. δ is adjoint of d; that is, $(\delta\omega_1, \omega_2) = (\omega_1, \mathrm{d}\omega_2)$.
cf. differential

coframe field Let M be a Riemannian manifold of dimension n and U a neighborhood of $m \in M$. Let $e_1, \ldots, e_n$ be a local orthogonal frame field on U and let $\omega_1, \ldots, \omega_n$ be the dual one-forms; that is,

$$\omega_i(e_j) = \delta_{ij} \ \text{ on } U.$$

Then $\omega_1, \ldots, \omega_n$ form a *local orthogonal coframe field on* U.

cohomology group *refer to* cochain complex

cohomology theory Given an admissible class $\mathcal{A}$ of pairs of spaces (X, A) and an abelian group G, a *cohomology theory on* $\mathcal{A}$ *with coefficients in* G consists of the following:

i. A function defined for each integer p and each pair (X, A) in $\mathcal{A}$ whose value is an abelian group $H^p(X, A; G)$.

ii. A function that for each integer p, assigns to each continuous map $h : (X, A) \longrightarrow (Y, B)$, a homomorphism

$$H^p(X, A; G) \xleftarrow{h^\star} H^p(Y, B; G).$$

iii. A function that for each integer p, assigns to each pair (X, A) in $\mathcal{A}$, a homomorphism

$$H^p(X, A; G) \xleftarrow{\delta^\star} H^{p-1}(A; G).$$

The following axioms are to be satisfied:

Axiom 1 If i is the identity map, then $i^\star$ is the identity.

Axiom 2 $(k \circ h)^\star = h^\star \circ k^\star$.

Axiom 3 $\delta^\star$ is a natural transformation of functors.

Axiom 4 The following sequence is exact, where i and j are inclusions:

$$\cdots \longleftarrow H^p(A; G) \xleftarrow{i^\star} H^p(X; G) \xleftarrow{j^\star} H^p(X, A; G) \xleftarrow{\delta^\star} H^{p+1}(A; G) \longleftarrow \cdots$$

Axiom 5 If h and k are homotopic, then $h^\star = k^\star$.

Axiom 6 Given (X, A), let U be an open set in X such that $\overline{U} \subset \operatorname{int} A$. If $(X - U, A - U)$ is admissible, then inclusion j induces a cohomology isomorphism

$$H^p(X - U, A - U; G) \xleftarrow{j^\star} H^p(X, A; G).$$

Axiom 7 If P is an one-point space, then $H^p(P; G) = 0$ for $p \neq 0$ and

$$H^0(P; G) \simeq G.$$

co-index Let M be a semi-Riemannian submanifold of $\overline{M}$. The index of $T_pM^\perp$ is called the *co-index of M in $\overline{M}$*.

cokernel of a function *refer to* kernel of a function

coline *refer to* future coray

collision A *collision in a Minkowski spacetime M* is a collection of r incoming material or lightlike particles:

$$\alpha_i : [a_i, 0] \longrightarrow M \quad (1 \leq i \leq r)$$

and s outgoing particles:

$$\beta_j : [0, b_j] \longrightarrow M \quad (1 \leq j \leq s)$$

such that $\alpha_i(0) = \beta_j(0) = p \in M$ for all i, j. Then p is called the *collision event*.

collision event *refer to* collision

commutative diagram A diagram of maps such as

$$\begin{array}{ccc} & & A \\ & \swarrow f & \downarrow h \\ B & \xrightarrow{g} & C \end{array}$$

is called *commutative* if $g \circ f = h$.

compact set *refer to* compact space

compact space Let S be a topological space. Then S is called *compact* if for every covering of S by open sets U_α (that is, $\cup U_\alpha = S$), there is a finite subcovering. A subset $A \subset S$ is called *cpt* if A is compact in the relative topology.

compact support The mapping $f \in \mathcal{F}(M)$ has *compact support* if $\mathsf{supp} f$ is compact in M.

complement of a set *refer to* topological space

complete atlas An atlas $\mathcal{C}$ on topological space $\mathcal{S}$ is *complete* if $\mathcal{C}$ contains each coordinate system in $\mathcal{S}$ that overlaps smoothly with every coordinate system in $\mathcal{C}$.
facta. Any atlas $\mathcal{A}$ on a Housdorff space makes it a manifold since we agree always to use the unique complete atlas containing $\mathcal{A}$.

completely integrable distribution *refer to* involutive distribution

complete solution of vector field Let X be a vector field on a n-manifold M. A *complete solution of* X is a triple (V, b, Ψ) where $V \subset M$ is an open set, $b \in \mathbf{R}$, $b > 0$ or $b = +\infty$, $I_b = (-b, b)$ and

$$\Psi : V \times I_b \longrightarrow \mathbf{R}^n$$

such that if $\Psi(u_0, 0) = c \in \mathbf{R}^n$, then

$$\{u \in V : \Psi(u, t) = c\}$$

is an integral curve of X at u_0. The components functions of a complete solution

$$\Psi(u, t) = \big(\psi_1(u, t), \ldots, \psi_n(u, t)\big)$$

are known as a *complete system of integrals of* X in the domain V.

complete system of integrals of X *refer to* complete solution of vector field

complete vector field A vector field V is *complete* provided its maximal integral curves are all defined on the whole real line.

complex projective space Let's introduce an equivalence relation in the complex n-sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$ by defining

$$(z_1, \ldots, z_{n+1}, 0, \ldots) \sim (\lambda z_1, \ldots, \lambda z_{n+1}, 0, \ldots)$$

for each complex number λ with $|\lambda| = 1$. The resulting quotient space is called *complex projective n-space* and is denoted by $\mathbf{C}P^n$.
facta. $\mathbf{C}P^n$ is Hausdorff.

component of a space *refer to* connected space

$\mathsf{Con}(M)$ Let $\mathsf{Con}(M)$ denote the quotient space formed by identifying all pointwise globally conformal metrics.

cone of a complex Let K be a complex in a generalized Euclidean space E^I and w a point of E^I such that each ray emanating from w intersects $|K|$ in at most one point. The *cone on K with vertex w* is the collection of all simplices of the form $wa_0 \cdots a_p$, where $a_0 \cdots a_p$ is a simplex of K along with all faces of such simplices. This collection is denoted by $w * K$. Such K is often called the *base of the cone*.

configuration space of a system *refer to* mechanical system with symmetry

conformally stable A property on $\mathsf{Lor}(M)$ is called *conformally stable* if it holds for an open set of equivalence classes in the quotient (or interval) topology on $\mathsf{Con}(M)$.

conformal mapping A smooth mapping $\varphi : M \longrightarrow N$ of semi-Riemannian mappings is *conformal* provided

$$\varphi^\star(\mathbf{g}_N) = h\mathbf{g}_M,$$

for some function $h \in \mathcal{F}(M)$ such that $h > 0$ or $h < 0$.
facta. If h is constant, φ is a homothety.

congruence *refer to* pair isometry

conjugate points Points $\sigma(a)$ and $\sigma(b)$, $a \neq b$, on a geodesic σ are *conjugate along σ* provided there is a nonzero Jacobi field J on σ such that $J(a) = 0$ and $J(b) = 0$.

connected set *refer to* connected space

connected space A topological space S is *connected* if ϕ and S are the only subsets of S that are both open and closed. A subset of S is *connected* if it is connected in the relative topology. A *component* A of S is a nonempty connected subset of S such that the only connected subset of S containing A is A. S is called *locally connected* if each point p has an open neighborhood containing a connected neighborhood of p.

connected sum If M and N are connected n-manifolds, their *connected sum* $M\#N$ is obtained by removing an open n-ball from each of M and N, and pasting the remnants together along their boundaries.

connection A *connection D on a smooth manifold M* is a function D from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to $\mathfrak{X}(M)$ such that

i. $D_V W$ is $\mathcal{F}(M)$-linear in V,
ii. $D_V W$ is **R**-linear in W,
iii. $D_V(fW) = (Vf)W + fD_V W$ for $f \in \mathcal{F}(M)$.

facta. $D_V W$ is called the covariant derivative of W with respect to V for the connection D.

constant curvature manifold A semi-Riemannian manifold M has *constant curvature* if the sectional curvature function is constant .

contact manifold A *contact manifold* is a pair (M, ω) consisting of an odd-dimensional manifold M and a closed two-form ω of maximal rank on M. An *exact contact manifold* (M, θ) consists of a $(2n+1)$-dimensional manifold M and an one-form θ on M such that $\theta \wedge (\mathrm{d}\theta)^n$ is a volume on M.

continuous curve in a semi-Riemannian manifold A future pointing causal curve $\gamma : (a, b) \longrightarrow M$ is said to be *continuous* if for each $t \in (a, b)$, there is an $\epsilon > 0$ and a convex normal neighborhood $U(\gamma(t))$ of $\gamma(t)$ with $\gamma(t-\epsilon, t+\epsilon) \subset U(\gamma(t))$ such that given any t_1, t_2 with $t-\epsilon < t_1 < t_2 < t+\epsilon$, there is a smooth future pointing causal curve in $U(\gamma(t))$ from $\gamma(t_1)$ to $\gamma(t_2)$.

contractible loop *refer to* loop

contractible space Topological space X is called *contractible* if X has the homotopy type of a single point.
rel. homotopy equivalence

contraction Note that there is a unique $\mathcal{F}(M)$-linear function

$$\mathsf{C} : \mathcal{T}^1_1(M) \longrightarrow \mathcal{F}(M),$$

called *(1,1) contraction*, such that $\mathsf{C}(X \otimes \theta) = \theta X$ for all $X \in \mathfrak{X}(M)$ and $\theta \in \mathfrak{X}^\star(M)$.
For $A \in \mathcal{T}^r_s(M)$, the function

$$(\theta, X) \longrightarrow A(\theta^1, \ldots, \theta, \ldots, \theta^{r-1}, X_1, \ldots, X, \ldots, X_{s-1})$$

is a (1,1) tensor that can be written

$$A(\theta^1, \ldots, \ , \ldots, \theta^{r-1}, X_1, \ldots, \ , \ldots, X_{s-1}).$$

Applying the (1,1) contraction to this tensor produces a real-valued function denoted by

$$(\mathsf{C}^i_j A)(\theta^1, \ldots, \theta^{r-1}, X_1, \ldots, X_{s-1}),$$

for $1 \leq i \leq r$ and $1 \leq j \leq s$. Then $\mathsf{C}^i_j A$ is $\mathcal{F}(M)$-multilinear and is a tensor of type (r-1, s-1) called the *contraction of A over i, j*.

contravariant functor A *contravariant functor G from a categry* C *to a category* D is a rule that assigns to each object X of C, an object $G(X)$ of D, and to each morphism $f : X \longrightarrow Y$ of C, a morphism $G(f) : G(X) \longrightarrow G(Y)$ of D such that

$$\begin{aligned} G(\mathrm{id}_X) &= \mathrm{id}_{G(X)} \text{ for all } X, \\ G(g \circ f) &= G(f) \circ G(g). \end{aligned}$$

A natural transformation between contravariant functors would be deined obviously.
cf. covariant functor

convergence Let Q be a semi-Riemannian manifold of M with mean curvature vector field H. The *convergence of Q* is the real-valued function $\mathbf{k}$ on the normal bundle NQ such that

$$\mathbf{k}(z) = \langle z, H_p \rangle = \frac{\mathsf{trace} S_z}{\dim N}$$

for $z \in T_pQ^{\perp}$. For spacelike hypersurface in M^n,

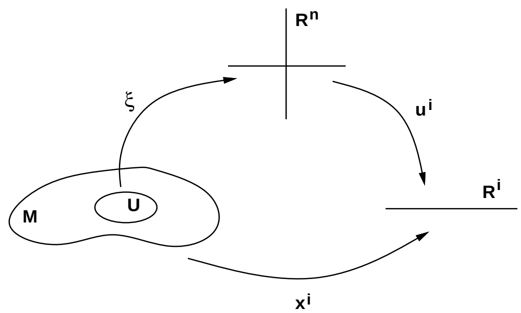
$$H_p = \frac{1}{n-1} \sum_{i=1}^{n-1} II(e_i, e_i),$$

where $e_1, \ldots, e_{n-1}$ is any orthonormal basis for T_pQ.
rel. focal point

convex covering A *convex covering $\mathcal{K}$ of a semi-Riemannian manifold M* is a covering of M by convex open sets such that if elements $\mathcal{U}, \mathcal{V}$ of $\mathcal{K}$ meet then $\mathcal{U} \cap \mathcal{V}$ is convex.
facta. For any open covering $\mathcal{O}$ of a manifold, there is a convex covering $\mathcal{K}$ such that each element of $\mathcal{K}$ is contained in some element of $\mathcal{O}$.

convex semi-Riemannian manifold An open set $\mathcal{C}$ in a semi-Riemannian manifold is *convex* provided $\mathcal{C}$ is a normal neighborhood of each of its points .

convex set A subset of $\mathbf{R}^n$ is called *convex* if for each pair x, y of points in A, the line segment joining them lies in A.



coordinate expression A *coordinate expression for f in terms of ξ* is $f \circ \xi^{-1} : \xi(U) \longrightarrow \mathbf{R}$. In fact,

$$f = (f \circ \xi^{-1})(x^1, \ldots, x^n) \quad \text{on} \quad U.$$

coordinate function *refer to* coordinate system

coordinate slice If $\xi : U \longrightarrow \mathbf{R}^n$ is a coordinate system in a manifold M then holding any $n - m$ of the coordinate functions constant produces an m-dimensional submanifold called a *ξ-coordinate slice* Σ of U.

coordinate system (chart) A *coordinate system (chart)* in a topological space $\mathcal{S}$ is a homeomorphism ξ of an open set U of $\mathcal{S}$ onto an open set $\xi(U)$ of $\mathbf{R}^n$. If we write

$$\xi(p) = (x^1(p), \ldots, x^n(p)) \quad \text{for each} \quad p \in U,$$

the resulting functions $x^1, \ldots, x^n$ are called *coordinate functions of ξ*. Thus

$$\xi = (x^1, \ldots, x^n) : U \longrightarrow \mathbf{R}^n$$

and $u^i \circ \xi = x^i$ where u^i is a natural coordinate function of $\mathbf{R}^n$.

coray *refer to* future coray

coset manifold If H is a closed subgroup of G, there is a unique way to make G/H a manifold so that the projection $\pi : G \longrightarrow G/H$ is a submersion. Such G/H is called a *coset manifold*.

cospacelike geodesic A geodesic σ in a manifold M is *cospacelike* provided the subspace $\sigma'(s)^{\perp}$ of $T_{\sigma(s)}M$ is spacelike for one (hence every) s.

cotangent bundle For manifold M, let *cotangent bundle $T^{\star}M$ of M* be the set $\bigcup\{T^{\star}_pM : p \in M\}$ of all cotangent spaces to M.
cf. tangent bundle

cotangent space The *cotangent space* of M at p is the set of all line maps of T_pM into **R** and denoted by $T_p^\star M$.

covariant derivative Let V be a vector field on a semi-Riemannian manifold M. The *(Levi-Civita) covariant derivative* D_V is the unique tensor derivation on M such that

$$D_V f = Vf \text{ for } f \in \mathcal{F}(M)$$

and $D_V W$ is the *Levi-Civita covariant derivative* for all $W \in \mathfrak{X}(M)$.

covariant differential The *covariant differential of an (r,s) tensor* A *on* M is the (r,s+1) tensor DA such that

$$(DA)(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s, V) = (D_V A)(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s)$$

for all $V, X_i \in \mathfrak{X}(M)$ and $\theta^j \in \mathfrak{X}^\star(M)$.

covariant functor A *covariant functor* G *from a categry* C *to a category* D is a function assigning to each object X of C, an object $G(X)$ of D, and to each morphism $f : X \longrightarrow Y$ of C, a morphism $G(f) : G(X) \longrightarrow G(Y)$ of D. The following two conditions must be satisfied:

$$\begin{aligned} G(1_X) &= 1_{G(X)} \text{ for all } X, \\ G(g \circ f) &= G(g) \circ G(f). \end{aligned}$$

That is, a covariant functor must preserve composition and identities. It is imediate that if f is an equivalence in$\mathsf{C}$, then $G(f)$ is an equivalence in D.
Simply, it is often called a *functor*.
cf. contravariant functor

cover *refer to* paracompact space

covering manifold *refer to* covering map

covering map A smooth map $k : \widetilde{M} \longrightarrow M$ onto M is a *covering map* provided each point $p \in M$ has a connected neighborhood U that is *evenly covered* by k; that is, k maps each component of $k^{-1}(U)$ diffeomorphically onto U. And the number of points in $k^{-1}(p)$ is called the *multiplicity* of the covering and $\widetilde{M}$is called a *covering manifold* .

critical energy density In Robertson-Walker spacetime , the *critical energy density* is expressed by $\rho_c = 3(H_0)^2/8\pi$, where Hubble number $H_0 = f_0'/f_0$.

critical manifold Let $N \subset M$ be a submanifold and each point in N is a critical point of f. N is called a *nondegenerate critical submanifold* if for each $p \in N$,

$$\left\{ v \in T_pM : H^f(v, w)|_p = 0 \text{ for all } w \in T_pM \right\} = T_pN.$$

facta. It is introduced by Bott in *Nondegenerate critical manifold*, Ann. Math. **60(2)** 248 – 261.

critical point A point $p \in M$ is a *critical point of* $f \in \mathcal{F}(M)$ provided $v(f) = 0$ for all $v \in T_pM$. It is often called a *singular ponit* or an *equilibrium point*.

cubic coordinate system A coordinate system φ of an open set U is called a *cubic coordinate system* if $\varphi(U)$ is an open cube about the origin in $\mathbf{R}^n$.

curl The *curl of* $V \in \mathfrak{X}(M)$ is defined

$$(\mathsf{curl}\, V)(X, Y) = \langle D_X V, Y\rangle - \langle D_Y V, X\rangle.$$

facta. 1. The $\mathsf{curl}\, V$ is a skew-symmetric (0,2) tensor field with coordinate components $\frac{\partial V_j}{\partial x^i} - \frac{\partial V_i}{\partial x^j}$.
2. $\mathsf{curl}\,(\mathsf{grad}\, f) = 0$.
3. $\mathsf{curl}\, V = \mathrm{d}\theta$, where θ is the one-form metrically equivalent to V.

curvaturelike function A multilinear function $F : T_pM^4 \longrightarrow \mathbf{R}$ is *curvaturelike* provided for the function $(v, w, x, y) \longrightarrow \langle \mathcal{R}_{vw}x, y\rangle$, F has the following symmetries :

i. $\mathcal{R}_{xy} = -\mathcal{R}_{yx}$
ii. $\langle \mathcal{R}_{xy}v, w\rangle = -\langle \mathcal{R}_{xy}w, v\rangle$
iii. $\mathcal{R}_{xy}z + \mathcal{R}_{yz}x + \mathcal{R}_{zx}y = 0$
iv. $\langle \mathcal{R}_{xy}v, w\rangle = \langle \mathcal{R}_{vw}x, y\rangle$

curve A *curve* in a manifold M is a smooth mapping $\alpha : I \longrightarrow M$ where I is an open interval in the real line $\mathbf{R}$.

cycle in homology *refer to* chain complex

D

Dajczer and Nomizu's criteria Let V have indefinite scalar product **g**, and let b be a symmetric bilinear form on V with corresponding quadratic form q. The followings are equivalent:

i. $b = C\mathbf{g}$ for some $C \in \mathbf{R}$.
ii. $q = 0$ on numm vectors .
iii. $|q|$ is bounded on timelike unit vectors .
iv. $|q|$ is bounded on spacelike unit vectors .

deck transformation A *deck transformation of a covering* $k : \widetilde{M} \longrightarrow M$ is a diffeomorphism $\phi : \widetilde{M} \longrightarrow \widetilde{M}$ such that $k \circ \phi = k$.

$$\begin{tikzcd}
\widetilde{\mathbf{M}} \arrow[rr, "\phi"] \arrow[dr, "\mathbf{k}"'] & & \widetilde{\mathbf{M}} \arrow[dl, "\mathbf{k}"] \\
& \mathbf{M} &
\end{tikzcd}$$

facta. 1. The set of all deck transformations of a covering forms a group with composition of functions as the group operation.
2. A deck transformation of a connected covering is determined by its value at a single point .
3. The more symmetrical a covering is, the larger its deck transformation group .

definite form A symmetric bilinear form b on V is

i. *positive* [*negative*] *definite* if $v \neq 0$ implies $b(v, v) > 0$ [< 0],
ii. *positive* [*negative*] *semidefinite* if for all $v \in V$, $b(v, v) \geq 0$ [≤ 0],
iii. *nondegenerate* if $b(v, w) = 0$ for all $w \in V$ implies $v = 0$.

deformation retract *refer to* deformation retraction

deformation retraction A *deformation retraction of X onto A* is a continuous map $F : X \times I \longrightarrow X$ such that

$$\begin{aligned} F(x, 0) &= x \qquad \text{for } x \in X, \\ F(x, 1) &\in A \qquad \text{for } x \in X, \\ F(a, t) &= a \qquad \text{for } a \in A. \end{aligned}$$

If such an F exists, then A is called a *deformation retract of X*.

degree of antipodal map *refer to* antipodal map

dense subset Let S be a topological space. A subset A of S is called *dense* in S if $\overline{A} = S$. And A is called *nowhere dense* if $\mathcal{C}(\overline{A})$ is dense in S.

derivation The *derivation on* $\mathcal{F}(M)$ is a function $\mathcal{D} : \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$ that is

i. **R**-linear : $\mathcal{D}(af + bg) = a\mathcal{D}(f) + b\mathcal{D}(g)$, for $a, b \in \mathbf{R}$.

ii. Leibnizian : $\mathcal{D}(fg) = g\mathcal{D}(f) + f\mathcal{D}(g)$, for $f, g \in \mathcal{F}(M)$.

derivation in exterior algebra *refer to* exterior algebra endomorphisms

de Sitter spacetime The four-dimensional Lorentz sphere $S_1^4(r)$ is called *de Sitter spacetime* .
cf. universal anti-de Sitter spacetime

diagonal map The *diagonal map* $d : X \longrightarrow X \times X$ *of* X is defined by $d(x) = (x, x)$.

diagonal set *refer to* Housdorff space

diameter For any metric space M, the *diameter of* M is defined by

$$\mathrm{diam}(M) = \sup \{d(p, q) : p, q \in M\},$$

where d is the Riemannian metric.

diffeomorphism A map $\phi : M \longrightarrow N$ is a *diffeomorphism* if ϕ is smooth and has an smooth inverse map.

differential The *differential of* $f \in \mathcal{F}(M)$ is the one-form $\mathrm{d}f$ such that $(\mathrm{d}f)(v) = v(f)$ for every tangent vector v to M.
facta. The differential has the following properties:

i. $\mathrm{d} : \mathcal{F}(M) \longrightarrow \mathfrak{X}^\star(M)$ is **R**-linear,

ii. $\mathrm{d}(fg) = g\mathrm{d}(f) + f\mathrm{d}(g)$, for $f, g \in \mathcal{F}(M)$,

iii. $\mathrm{d}(h(f)) = h'(f)\mathrm{d}f$, for $f \in \mathcal{F}(M)$, $h \in \mathcal{F}(\mathbf{R})$.

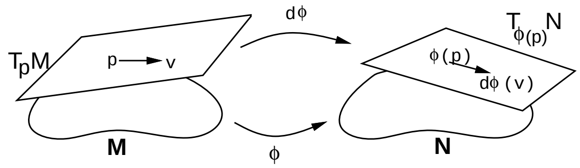
differential ideal Let $E^\star(M)$ be the set of all differential forms. An ideal $\mathcal{I} \subset E^\star(M)$ is called a *differential ideal* if it is closed under exterior differentiation d; that is,

$$\mathrm{d}(\mathcal{I}) \subset \mathcal{I}.$$

differential map Let $\phi : M \longrightarrow N$ be a smooth mapping. For each $p \in M$,

$$\mathrm{d}\phi_p : T_pM \longrightarrow T_{\phi(p)}N$$

sending v to v_ϕ is called the *differential map of* ϕ *at* p.
facta. 1. $\mathrm{d}\phi_p(v)(g) = v(g \circ \phi)$, for all $v \in T_pM$ and $g \in \mathcal{F}(M)$.
2. differential maps are linear.



differential p-form A *differential p-form* is a skew-symmetric covariant tensor field of type $(0, p)$. Let $U \subset M$ be open. The set of differential p-forms on U forms a real vector space and is denoted by $E^p(U)$.

Dimension axiom *refer to* homology theory

discrete topology Let S be a topological space. A point $u \in S$ is called *isolated* if $\{u\}$ is open in S. The unique topology in which every point is isolated is called the *discrete topology*; $\mathcal{O} = 2^S$, the collection of all subsets.

distance preserving *refer to* homothety

distinguishing spacetime A spacetime is said to be *distinguishing* if for all points $p, q \in M$, either $I^+(p) = I^+(q)$ or $I^-(p) = I^-(q)$ implies $p = q$.
facta. In a distinguishing spacetime , distinct points have distinct chronological futures and chronological pasts. Thus points are distinguished both by their chronological futures and pasts.

distribution Let c be an integer with $1 \leq c \leq d$. A *c-dimensional distribution $\mathcal{D}$* on a d-dimensional manifold M is a choice of a c-dimensional subspace $\mathcal{D}(m)$ of T_mM for each m in M. $\mathcal{D}$ is *smooth* if for each m in M, there is a neighborhood U of m and there are c smooth vector fields $X_1, \ldots, X_c$ which span $\mathcal{D}$ at each point of U. A vector field X on M is said to *belong to* (or *lying in*) *the distribution* $\mathcal{D}$ if $T_mX \in \mathcal{D}(m)$ for each $m \in M$.

divergence Let M be an orientable manifold with volume element ω and X a vector field on M. Then the unique function $\mathsf{div}X \in \mathcal{F}(M)$ such that $L_X\omega = (\mathsf{div}X)\omega$ is called the *divergence of* X.

divergence theorem Let V be a smooth vector field on an oriented Riemannian manifold M and let D be a regular domain in M. If $\vec{n}$ is the unit normal vector field on ∂D, then

$$\int_D \mathsf{div}V = \int_{\partial D} \langle V, \vec{n} \rangle.$$

dual homomorphism A homomorphism $f : A \longrightarrow B$ give rise to a *dual homomorphism* $\tilde{f}$ such that

$$\mathsf{Hom}\ (A, G) \xleftarrow{\tilde{f}} \mathsf{Hom}\ (B, G)$$

giving in the reverse direction. The map $\tilde{f}$ assigns the homomorphism $\phi : B \longrightarrow G$ to the composite map

$$A \xrightarrow{f} B \xrightarrow{\phi} C.$$

That is, $\tilde{f}(\phi) = \phi \circ f$.

dual normal symmetric space Normal symmetric spaces $M = G/H$ and $M^\star = G^\star/H^\star$ are *dual* provided there exist

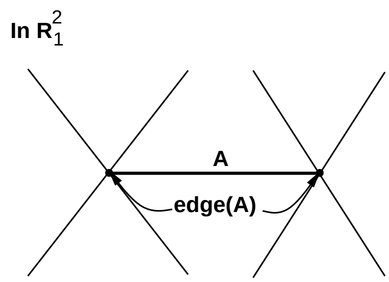
i. a Lie algebra isomorphism $\tilde{\delta} : \mathsf{h} \longrightarrow \mathsf{h}^\star$ such that $B^\star(\tilde{\delta}V, \tilde{\delta}W) = -B(V, W)$ for all $V, W \in \mathsf{h}$;

ii. a linear isometry $\delta : \mathsf{m} \longrightarrow \mathsf{m}^\star$ such that $[\delta X, \delta Y] = -\tilde{\delta}[X, Y]$ for all $X, Y \in \mathsf{m}$.

dust A *dust* is a perfect fluid with pressure $p = 0$ and energy density $\rho > 0$.

E

edge The *edge of an achronal set* A consists of all points $p \in \overline{A}$ such that every neighborhood U of p contains a timelike curve from $I^-(p,U)$ to $I^+(p,A)$ that does not meet A.



Eilenberg-Steenrod axioms *refer to* homology theory

Eilenberg-Zilber theorem For any pair X, Y of topological spaces, there are chain maps between singular chain complexes

$$\mathcal{S}(X) \otimes \mathcal{S}(Y) \rightleftharpoons \mathcal{S}(X \times Y)$$

that are chain-homotopy inverse to each other; they are natural with respect to chain maps induced by continuous maps.

Einstein-de Sitter cosmological model The *Einstein-de Sitter cosmological model* is

$$M(0, t^{\frac{2}{3}}) = \mathbf{R}^+ \times_{t^{2/3}} \mathbf{R}^3.$$

facta. The Hubble function $H = f'/f = \frac{2}{3t}$.

Einstein field equation If M is a spacetime containing matter with stress-energy tensor T, then

$$G = 8\pi T,$$

where G is the Einstein gravitational tensor .
facta. The *universal constant* $k = 8\pi$ is determined from comparison with Newtonian physics at low speeds and weak gravitation.

Einstein gravitational tensor The *Einstein gravitational tensor of a spacetime* M is $G = \mathsf{Ric} - \frac{1}{2}S\mathbf{g}$.
facta. 1. G is a symmetric (0,2) tensor field with divergence zero.
2. $\mathsf{Ric} = G - \frac{1}{2}\mathsf{C}(G)\mathbf{g}$.

Einstein manifold A semi-Riemannian manifold M is an *Einstein manifold* provided $\mathsf{Ric} = c\mathbf{g}$ for some constant c.
facta. If M is connected, $\dim M \geq 3$ and $\mathsf{Ric} = f\mathbf{g}$, then M is *Einstein*.

elliptic operator Let L be a partial differential operator of order l and D the derivative operator. Let

$$L = P_l(D) + \cdots + P_0(D),$$

where $P_j(D)$ is an $m \times m$ matrix each entry of which is a differential operator $\sum_{[\alpha]=j} a_\alpha D^\alpha$, homogeneous of order j and where the a_α are smooth complex valued functions on $\mathbf{R}^n$. Let $P_j(\xi)$ denote the matrix obtained by substituting ξ^α for D^α in $P_j(D)$ where $\xi = (\xi 1, \ldots, \xi n)$ is a point in $\mathbf{R}^n$. L is said to be *elliptic at the point* $x \in \mathbf{R}^n$ if the matrix $P_l(\xi)$ is nonsingular at x for each nonzero ξ. L is *elliptic* if it is elliptic at each x.
facta. L is elliptic at x if and only if

$$L(\varphi^l u)(x) \neq 0,$$

for each $\mathbf{C}^m$-valued smooth function u such that $u(x) \neq 0$ and each smooth real-valued function φ such that $\varphi(x) = 0$ but $\mathrm{d}\varphi(x) \neq 0$, since for each such φ and u,

$$L(\varphi^l u)(x) = P_l(D)(\varphi^l u)(x) = P_l(\mathrm{d}\varphi|_x)(u(x)).$$

embedding *refer to* imbedding

energy For a curve segment $\alpha : [0, b] \longrightarrow M$ in a semi-Riemannian manifold, the integral

$$E(\alpha) = \frac{1}{2}\int_0^b \langle \alpha', \alpha' \rangle \mathrm{d}u$$

is called *energy* (or *action*) of a curve α. For a piecewise smooth variation $\mathbf{x}$ of α, let $E_{\mathbf{x}}(v)$ be the value of E on the longitudinal curve $v \rightarrow \mathbf{x}(u, v)$. So

$$E_{\mathbf{x}}(v) = \frac{1}{2}\int_0^b \langle \mathbf{x}_u, \mathbf{x}_u \rangle \mathrm{d}u.$$

energy equation If (U, ρ, p) is a perfect fluid,

$$U\rho = -(\rho + p)\mathsf{div}U$$

is the *energy equation* .
rel. stress-energy tensor

energy equation in Schwarzschild spacetime Let N be the Schwarzschild exterior and B black hole. If γ is a lightlike particle in $N \cup B$, then relative to equatorial coordinates ,

i. $h\mathrm{d}t/\mathrm{d}s = E$
ii. $r^2\mathrm{d}\varphi/\mathrm{d}s = L$
iii. $\vartheta = \pi/2$

and γ satisfies the *energy equation*

$$E^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \left(\frac{L^2}{r^2}\right) h(r).$$

facta. On N we interpret the constant E and L as the energy at infinity and angular momentum of γ.

energy in Minkowski spacetime Let α be a material particle of mass m in Minkowski spacetime M. If ω is freely falling observer, then the *energy of α relative to ω* is the time component

$$E = \frac{m}{\sqrt{1 - v^2}}$$

of $P = m\mathrm{d}\alpha/\mathrm{d}\tau$, and the *momentum of α relative to ω* is the Euclidean vector field

$$P = \frac{m}{\sqrt{1 - v^2}} \frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t}$$

on $\tilde{\alpha}$. The *scalar momentum of α relative to ω* is the function $\wp = |\widetilde{P}|$.

energy-momentum vector field Let M be a Minkowski spacetime. If $\alpha : I \longrightarrow M$ is a material particle of mass m, its *energy-momentum vector field* is the vector field $P = m\mathrm{d}\alpha/\mathrm{d}\tau$ on α.

ε-disk *refer to* metric

equilibrium point *refer to* critical point

equivalence class *refer to* equivalence relation

equivalence relation Let S be a set. An *equivalence relation* $\sim$ on S is a binary relation such that for all $u, v, w \in S$,

i. $u \sim u$ (*reflexive law*);
ii. $u \sim v$ iff $v \sim u$ (*symmetric law*);
iii. $u \sim v$ and $v \sim w$ implies $u \sim w$ (*transitive law*).

The *equivalence class* containing u, denoted $[u]$, is defined by

$$[u] = \{v \in S : u \sim v\}.$$

The set of equivalence class is denoted $S/\sim$ and the mapping $\pi : S \longrightarrow S/\sim$ with $u \mapsto [u]$ is called the *canonical projection*.

Euclidean coordinate system An *Euclidean coordinate system for E* is an isometry $\zeta : E \longrightarrow \mathbf{R}^3$.

ergodic flow Let S be a measure space and ψ_t a flow on S. If for all t, the only invariant measurable sets under ψ_t are $\emptyset$ and all of S, then ψ_t is called *ergodic*.

Euclidean half-space *Euclidean half-space H^n* is defined by

$$H^n = \{ (x_1, \dots, x_n) | \, x_n \geq 0 \} .$$

Euclidean metric The *Euclidean metric on* $\mathbf{R}^n$ is defined by

$$d(x, y) = \left(\sum_{i=1}^{n} (x^i - y^i)^2 \right)^{\frac{1}{2}} ,$$

where $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^n)$. The above metric is often called the *standard metric* on $\mathbf{R}^n$.

Euclidean n-space On $\mathbf{R}^n$, the dot product gives rise to a metric tensor with

$$\langle v_p, w_p \rangle = v \cdot w = \sum_i v^i w^i ,$$

for $v, w \in \mathbf{R}^n$. In any geometric context $\mathbf{R}^n$ will denote the resulting Riemannian manifold , called *Euclidean n-space*.

Euler number The *Euler number of a finite complex K* is defined by

$$\chi(K) = \sum_p (-1)^p \mathsf{rank}\ (C_p(K)).$$

That is, $\chi(K)$ is the alternating sum of the number of simplices of K in each dimension.
facta. Euler number of K is a topological invariant of $|K|$.

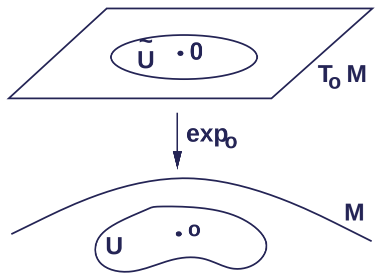
Euler-Poincaré characteristic Given a triangulation $\mathcal{T}$ of a regular region $R \subset S$ of a surface S, we may denote by F the number of triangles (faces), by E the number of edges and by V the number of vertices of the triangulation. The number

$$F - E + V = \chi$$

is called the *Euler-Poincaré characteristic of the triangulation*.

evenly covered *refer to* covering map

event In a Minkowski spacetime M, every point in M is called an *event*.



event horizon Let A be a subset of a connected time-oriented Lorentz manifold. The *event horizon* $\mathcal{E}$ is the boundary of $J^+(A)$; that is, $\mathcal{E} = \text{bd}J^+(A)$.

exact contact manifold *refer to* contact manifold

exact form Let A be open in M. A 0-form f on A is said to be *exact* on A if it is constant on A; a k-form $omega$ on A with $k > 0$ is said to be *exact on* A if there is a $(k-1)$-form θ on A such that $\omega = \text{d}\theta$.

Exactness axiom *refer to* homology theory

exact sequence Consider a sequence (finite or infinite) of groups and homomorphisms

$$\cdots \longrightarrow A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \longrightarrow \cdots$$

This equation is called *exact at* A_2 if

$$\text{im}\phi_1 = \text{ker}\phi_2.$$

If it is everywhere exact, it is called an *exact sequence*. Of course, exactness is not defined at the first or last group of sequence if such exist.
cf. closed form

Excision axiom *refer to* homology theory

exponential map If $o \in M$, let $\mathcal{E}_o$ be the set of vectors v in T_oM such that the inextendible geodesic γ_n is defined at least on [0,1]. The *exponential map of* M *at* o is the function

$$\exp_o : \mathcal{E}_o \longrightarrow M$$

such that $\exp_o(v) = \gamma_v(1)$ for all $v \in \mathcal{E}_o$.
facta. The exponential map $\exp_o$ carries lines through the origin of T_oM to geodesics of M through o.

extendible curve A piecewise smooth curve $\alpha : [0, B) \longrightarrow M$ is *extendible* provided that it has a continuous extension $\tilde{\alpha} : [0, B] \longrightarrow M$. Then $q = \tilde{\alpha}(B)$ is called an *endpoint* of α.

extendible manifold A connected semi-Riemannian manifold M is *extendible* provided M is (isometric to) an open submanifold of a connected semi-Riemannian manifold $\widetilde{M} \neq M$. In general, M is *extendible* if one of its connected components is extendible. Otherwise M is *inextendible* (or *maximal*).
rel. extension of a manifold

extension of a manifold An *extension of a Lorentz manifold* M is a Lorentz manifold $\widetilde{M}$ together with an isometry $\psi : M \longrightarrow \widetilde{M}$ which maps M onto a proper open subset of $\widetilde{M}$. An *analytic extension of* M is an extension $\psi : M \longrightarrow \widetilde{M}$ such that both Lorentz manifolds are analytic and the map ψ is analytic.
rel. extendible manifold

exterior algebra Let $\Lambda^k(V)$ be the vector space of skew-symmetric k-multilinear maps on a finite dimensional real vector space V. Then the maps in $\Lambda^k(V)$ are called *exterior k-forms on* V. For $\omega_1 \in \Lambda^k(V)$ and $\omega_2 \in \Lambda^l(V)$, the wedge product $\omega_1 \wedge \omega_2 \in \Lambda^{k+l}(V)$ is defined by

$$\omega_1 \wedge \omega_2 = \frac{(k+l)!}{k!l!} h(\omega_1 \otimes \omega_2),$$

for an alternating multilinear map h. The direct sum of the spaces $\Lambda^k(V)$ $(k = 0, 1, \ldots)$ together with its structure as a real vector space and multiplication induced by the wedge product, is called the *exterior algebra of* V or the *Grassmann algebra of* V and is denoted by $\Lambda(V)$.

exterior algebra endomorphisms An *endomorphism* l *of an exterior algebra* $\Lambda(V)$ *of vector space* V (or, *of any graded algebra*) is

i. a derivation if $l(u \wedge v) = l(u) \wedge v + u \wedge l(v)$, for $u, v \in \Lambda(V)$;
ii. an anti-derivation if $l(u \wedge v) = l(u) \wedge v + (-1)^p u \wedge l(v)$, for $u \in \Lambda_p(V)$ and $v \in \Lambda(V)$;
iii. of degree k if $l : \Lambda_j(V) \longrightarrow \Lambda_{j+k}(V)$ for all j with assumption $\Lambda_i(V) = \{0\}$ if $i < 0$.

exterior k-form *refer to* exterior algebra

F

face of simplex *refer to* abstract simplicial complex

fiber *refer to* 1. vector bundle
2. warped product

fiber deivative let M be a manifold and $L \in \mathcal{F}(TM)$. Then the map $\mathcal{F}L : TM \longrightarrow T^{\star}M$ with $w_p \mapsto DL_p(w_p) \in L(T_pM, \mathbf{R}) = T_p^{\star}M$ is called the *fiber derivative of* L, where L_p denotes the restriction of L to the fiber over $p \in M$.

filtered space *refer to* filtration

filtration If X is a space, a *filtration of* X is a sequence $X_0 \subset X_1 \subset \cdots$ of subspaces of X whose union is X. A space X together with a filtration of X is called a *filtered space*. If X and Y are filtered spaces, a continuous map $f : X \longrightarrow Y$ such that for all p, $f(X_p) \subset Y_p$ is said to be *filtration-preserving*.

filtration-preserving *refer to* filtration

fine C^r topologies Let $\mathsf{Lor}(M)$ denote the space of all Lorentz metrics on M. The *fine C^r topologies on* $\mathsf{Lor}(M)$ may be defined by using a fixed countable covering $\mathcal{B} = \{B_i\}$ of M by coordinate neighborhoods with the property that each compact subset of M intersects only finitely many of the B_i's. (Such a coordinate covering is called *locally finite*.)

finite distance condition The spacetime $(M, \mathbf{g})$ satisfies the *finite distance condition* if $d(p, q) < \infty$ for all $p, q \in M$.

finitely compact spacetime The causal spacetime $(M, \mathbf{g})$ is called *finitely compact* if for each fixed constant $B > 0$ and each sequence of points $\{r_n\}$ with either $p \ll q \leq r_n$ and $d(p, r_n) \leq B$ for all n, or $r_n \leq q \ll p$ and $d(r_n, p) < B$ for all n, there is an accumulation point of $\{r_n\}$ in M.
facta. Without the condition $x_n \leq q \ll p$ (or $x_n \leq q \ll p$) for some $q \in M$, Minkowski spacetime fails to be finitely compact .

finitely generated group *refer to* free abelian group

first countable space *refer to* first countable topology

first countable topology Let S be a topological space. The topology is called *first countable* if for each $u \in S$, there is a countable collection $\{U_n\}$ of neighborhoods of u such that for any neighborhood U of u, there is an integer n with $U_n \subset U$. Such a space S is called a *first countable space*.

first fundamental form *refer to* second fundamental form

fixed-endpoint homotopic *refer to* fixed-endpoint homotopy

fixed-endpoint homotopy Let $P(p,q)$ be the set of all paths from p to q. If $\alpha, \beta \in P(p,q)$, a *fixed-endpoint homotopy from α to β* is a continuous map $H : I \times I \longrightarrow M$ such that for all $s, t \in I$

$$\begin{aligned} H(t,0) &= \alpha(t) \qquad & H(0,s) &= p, \\ H(t,1) &= \beta(t) \qquad & H(1,s) &= q. \end{aligned}$$

Defining $\alpha_s(t) = H(t,s)$ shows that H is an one-param family of path $\alpha_s \in P(p,q)$, varying continuously from $\alpha_0 = \alpha$ to $\alpha_1 = \beta$. If such a homotopy exists, α and β are *fixed-endpoint homotopic*, denoted by $\alpha \simeq \beta$.

fixed-endpoint homotopy class Since fixed-endpoint homotopy is an equivalence relation on $P(p,q)$, the equivalence class containing $\alpha \in P(p,q)$ is denoted by $[\alpha]$ and called the *fixed-endpoint homotopy class of α*.

flat manifold A semi-Riemannian manifold M for which the Riemannian curvature tensor $\mathcal{R}$ is zero at every point is said to be *flat*.

flat operator Let M be a manifold and $\omega \in \Lambda^2(M)$ be nondegenerate. Then the map $\flat : \mathfrak{X}(M) \longrightarrow \mathfrak{X}^\star(M)$ with $X \mapsto X^\flat = i_X\omega$ (hence $X^\flat = \omega^\flat(X)$) is called the *flat operator*. And the map $\sharp : \mathfrak{X}^\star(M) \longrightarrow \mathfrak{X}(M)$ with $\alpha \mapsto \alpha^\sharp = \omega^\sharp(\alpha)$ is called the *sharp operator*.
facta. $(X^\flat)^\sharp = X$ and $(\alpha^\sharp)^\flat = \alpha$.

flow The *flow of a complete vector field V on M* is the mapping $\psi : M \times \mathbf{R} \longrightarrow M$ is given by $\psi(p,t) = \alpha_p(t)$, where α_p is the maximal integral curve starting at p.
rel. flow box

flow box Let M be a manifold and X a vector field on M. A *flow box of X at $m \in M$* is a triple (U, a, ψ) where

i. $U \subset M$ is open with $m \in U$ and $a \in \mathbf{R}$, $a > 0$ or $a = +\infty$;
ii. $F : U \times I_a \longrightarrow M$ is a class C^∞ where $I_a = (-a, a)$;
iii. for each $u \in U$, $\gamma_n : I_a \longrightarrow M$ defined by $\gamma_n(\lambda) = \psi(u, \lambda)$ is an intergral curve of X at u;
iv. if $\psi_\lambda : U \longrightarrow M$ is defined by $\psi_\lambda(u) = \psi(u, \lambda)$, then for $\lambda \in I_a$, $\psi_\lambda(U)$ is open and ψ_λ is a diffeomorphism onto its image.

Such a mapping ψ is called the *flow of X*.
rel. flow
facta. 1. (*uniqueness of the flow box*) When (U, a, ψ) and (U', a', ψ') are two flow boxes at $m \in M$, ψ and ψ' are equal on $(U \cap U') \times (I_a \times I_{a'})$.
2. (*Existence of the flow box*) Let X be a smooth vector field on a manifold M. For each $m \in M$, there is a flow box of X at m.

focal order *refer to* focal point

focal point Let σ be a geodesic of M that is normal to $P \subset M$, that is $\sigma(0) \in P$, $\sigma'(0) \perp P$. Then $\sigma(r)$, $r \neq 0$, is a *focal point of P along σ* provided there is a nonzero P-Jacobi field J on σ with $J(r) = 0$. The *focal order* of $\sigma(r)$ is the dimension of the space of P-Jacobi fields on σ that vanish at r.

force equation If (U, ρ, p) is a perfect fluid, the *force equation* is

$$(\rho + p)D_U U = -\mathsf{grad}_\perp p,$$

where the *spatial pressure gradient* $\mathsf{grad}_\perp p$ is the component of $\mathsf{grad}\, p$ orthogonal to U.
rel. stress-energy tensor

frame An orthonormal basis for a tangent space T_pM is called a *frame on M at p*.

frame field A *frame field on a curve $\alpha : I \longrightarrow M$* is a set of mutually orthogonal unit vector field $E_1, \ldots, E_n$ on α.

frame-homogeneous A semi-Riemannian manifold M is *frame-homogeneous* provided any frame on M can be carried to any other by the differential map of an isometry of M.
facta. Hyperquadrics are frame-homogeneous.

free abelian group An abelian group G is *free* if it has a basis; that is, if there is a family $\{g_\alpha\}_{\alpha \in I}$ of elements of G such that each $g \in G$ can be written uniquely as a finite sum

$$g = \sum_\alpha n_\alpha g_\alpha$$

with an integer n_α. If each $g \in G$ can be written as a finite sum $g = \sum n_\alpha g_\alpha$ but not necessarily uniquely, then we say that the family $\{g_\alpha\}$ *generates* G. In particular, if the set $\{g_\alpha\}$ is finite, we say that G is *finitely generated*. The number of elements in a basis for G is called the *rank of G*.
facta. Uniqueness implies that each element g_α has infinite order; that is, g_α generates an infinite cyclic subgroup of G.

free falling A particle in a Minkowski space is a geodesic is said to be *free falling*. In general, “free falling” means moving under the influence of gravity alone.

free homotopy A homotopy of closed curves (loops) in which the endpoint is allowed to move is called a *free homotopy*.

Friedmann cosmological model A *Friedmann cosmological model* is a Robertson-Walker spacetime such that the galactic fluid is dust and $H = f'/f$ is positive for some t_0.

Friedmann equation Let $M(k, f)$ be a Robertson-Walker spacetime with f nonconstant. The *Friedman equation* is

$$f'^2 + k = A/f, \text{ where } \quad A = 8\pi\mathsf{M}/3.$$

Frobenius theorem Let $\mathcal{D}$ be a c-dimensional involutive smooth distribution on d-manifold M. Let $m \in M$. Then there exists an integral manifold of coordinate system (U, φ) which is centered at m. Indeed, there exists a cubic coordinate functions $x_1, \ldots, x_d$ such that the slices

$$x_i = \text{constant} \quad \text{for all } i \in \{c+1, \ldots, d\}$$

are integral manifolds of $\mathcal{D}$ and if (N, ψ) is a connected integral manifold of $\mathcal{D}$ such that $\psi(N) \subset U$, then $\psi(N)$ lies in one of these slices.

full quantization Let Q be a manifold. A *full quantization of* Q is a map taking classical observables f (i.e., continuous functions of $(q, p) \in T^\star\mathbf{R}^n$) to self-adjoint operators $\hat{f}$ on Hilbert space $\mathcal{H}$ such that

i. $(f+g)\hat{} = \hat{f} + \hat{g}$;
ii. $(\lambda f)\hat{} = \lambda\hat{f}$ for $\lambda \in \mathbf{R}$;
iii. $\{f, g\}\hat{} = \frac{1}{i}[\hat{f}, \hat{g}]$;
iv. $\hat{c} = \text{id}$ for constant function c;
v. $\hat{q}_i$ and $\hat{p}_j$ act irreducibly on $\mathcal{H}$.

facta. The condition (v) really means that we can take $\mathcal{H} = L^2(\mathbf{R}^n)$ and that $\hat{q}_i$ and $\hat{p}_j$ are given by $\hat{q}_i = Q_{q_i}$ and $\hat{p}_j = \frac{1}{i}\frac{\partial}{\partial q_j}$; it is called the *Schrödinger representation*.

full subcomplex Let L be a complex. A subcomplex L_0 of L is said to be a *full subcomplex of* L provided each simplex of L whose vertices belong to L_0 belongs to L_0 itself.

functor *refer to* covariant functor

fundamental group If $p \in M$, let $\pi_1(M, p)$ be the set of all fixed-endpoint homotopy classes in $P(p, p)$. The multiplication $[\alpha][\beta] = [\alpha * \beta]$ makes $\pi_1(M, p)$ a group, called the *fundamental group of* M *at* p, where $\alpha * \beta$ means the path product of α and β.
facta. Above property is first proposed by Poincaré.

fundamental inequality of elliptic operator Let $\mathcal{P}$ denote the complex vector space consisting of smooth functions defined on $\mathbf{R}^n$ which have values in complex m-space $\mathbf{C}^m$ and are periodic of period 2π ineach variable. Let L

be an elliptic operator on $\mathcal{P}$ of order l and let s be an integer. Then there is a constant $c > 0$ such that

$$\|u\|_{s+l} \leq c \left(\|Lu\|_l + \|u\|_s \right)$$

for all $u \in H_{s+l}$, where $\|u\|_k$ means the L^k norm of u.

Fundamental theorem of finitely generated abelian groups Let G be a finitely generated abelian group. Let T be its torsion subgroup.

i. There is a free abelian subgroup H of G having finite rank β such that $G = H \oplus T$.
ii. There are finite cyclic groups $T_1, \ldots, T_k$, where T_i has order $t_i > 1$ such that $t_1 | t_2 | \cdots | t_k$ and

$$T = T_1 \oplus \cdots \oplus T_k.$$

iii. The number β and $t_1, \ldots, t_k$ are uniquely determined by G.

The number β is called the *Betti number* og G and the numbers $t_1, \ldots, t_k$ are called the *torsion coefficients* of G.
facta. β is the rank of the free abelian group $G/T \simeq H$.

Fundamental theorem of Riemannian geometry Let M be a semi-Riemannian manifold. Then there is a unique connection D on M such that

i. $D_X Y - D_Y X = [X, Y]$;
ii. parallel translation preserves the inner product (i.e., is an isometry).

future Busemann function *refer to* Busemann function

future Cauchy development *refer to* Cauchy development

future Cauchy horizon *refer to* Cauchy horizon

future causal cut locus The *future causal cut locus* $C^+(p)$ *of* p is the union of future timelike cut locus and future null cut locus; that is, $C^+(p) = C_t^+(p) \cup C_N^+(p)$. The *past causal cut locus* $C^-(p)$ *of* p is defined dually; that is, $C^-(p) = C_t^-(p) \cup C_N^-(p)$.
The *causal cut locus* $C(p)$ *of* p is defined by $C(p) = C^-(p) \cup C^+(p)$.

future-converging A spacelike submanifold of M is *future-converging* provided its mean curvature vector field H is past pointing timelike .

future coray Let $p \in J(\gamma) \equiv J^+(\gamma) \cap J^-(\gamma)$. A future [past] ray $\mu_\pm : I_\pm \longrightarrow M$ from [to] p satisfying

i. $\mu_\pm \subset J(\gamma)$;

ii. $b^{\pm}_{\gamma}(\mu_{\pm}(v)) = b^{\pm}_{\gamma}(\mu_{\pm}(u)) + d(\mu_{\pm}(u), \mu_{\pm}(v))$
for all $u, v \in I_{\pm}$ such that $u \leq v$,

is called a *future [past] coray* of γ. A line $\eta : I \longrightarrow M$ satisfying

i. $\eta \subset J(\gamma)$;

ii. $b^{+}_{\gamma}(\eta(v)) = b^{-}_{\gamma}(\eta(v)) = b^{+}_{\gamma}(\eta(u)) + d(\eta(u), \eta(v))$
for all $u, v \in I$ such that $u \leq v$,

is called a *coline* of γ.

future cut point *refer to* future timelike cut locus

future-directed *refer to* future pointing

future imprisoned Let $\gamma : [a, b) \longrightarrow M$ be a future pointing causal curve. Then γ is called *future imprisoned in the compact set K* if there is some $t_0 < b$ such that $\gamma(t) \in K$ for all $t_0 < t < b$. The curve γ is called *partially future imprisoned in the compact set K* if there exists an infinite sequence $t_n \rightarrow b$ with $\gamma(t_n) \in K$ for each n.

future inner ball *refer to* inner ball

future null cut locus Let $C^{+}_{N}(p)$ [$C^{-}_{N}(p)$] denote the *future* [*past*] *null cut locus of* p which is consists of all future [past] null cut points of p.

future nullcone *refer to* nullcone

future null cut point Let $\gamma : [0, a) \longrightarrow M$ be a future pointing null geodesic with endpoint $p = \gamma(0)$. Let $t_0 = \sup\{t \in [0, a) : d(p, \gamma(t)) = 0\}$. If $0 < t_0 < a$, we will say $\gamma(t_0)$ is the *future null cut point of p on γ*.
The *past null cut points* are defined dually.

future outer ball *refer to* outer ball

future pointing A tangent vector in a future causal cone is said to be *future pointing* (or *future-directed*). A causal curve is *future pointing* if all its velocity vectors are future pointing.

future pre-Busemann function *refer to* pre-Busemann function

future set A subset F of M is a *future set* provided $I^{+}(F) \subset F$. If F is a future set, its complement $M - F$ is a *past set* (closed under I^{-}).

future timelike cut locus Let

$$T_{-1}M = \{v \in TM : \mathbf{g}(v, v) = -1 \text{ and } v \text{ is future pointing }\}.$$

Given $p \in M$, let $T_{-1}M|_p$ denote the fiber of $T_{-1}M$ at p. Also given $v \in T_{-1}M$, let σ_v denote the unique timelike geodesic with $\sigma_v'(0) = v$. The *future timelike cut locus* $\Gamma^+(p)$ *in* T_pM is defined to be

$$\Gamma^+(p) = \{s(v)v : v \in T_{-1}M|_p \text{ and } 0 < s(v) < \infty\}.$$

The *future timelike cut locus* $C_t^+(p)$ *of* p *in* M is defined by

$$C_t^+(p) = \exp_p(\Gamma^+(p)).$$

If $0 < s(v) < \infty$ and $\sigma_v(s(v))$ exists, then the point $\sigma_v(s(v))$ is called the *future cut point of* $p = \sigma_v(0)$ *along* σ_v.
The *past timelike cut locus* $C_t^-(p)$ and *past cut point* may be defined dually.

future-trapped A closed achronal subset A of M is *future-trapped* provided $E^+(A) \equiv J^+(A) - I^+(A)$ is compact.
Dually, *past-trapped* means $E^-(A) \equiv J^-(A) - I^-(A)$ is compact.

G

Gauss-Bonnet theorem Let $R \subset S$ be a regular region of an oriented Riemannian surface and let $C_1, \ldots, C_n$ be the closed, simple, piecewise regular curves which from the boundary bd$R$ of R. Let $\chi(R)$ denote the Euler-Poincaré characteristic of R, $k_g(s)$ the geodesic curvature of the regular arclength of α and K the Gaussian curvature of S. Suppose that each C_i is positively oriented and let $\theta_1, \ldots, \theta_n$ be the set of all external angles of the curves $C_1, \ldots, C_n$. Then

$$\sum_{i=1}^{n} \int_{C_i} k_g(s) \mathrm{d}s + \iint_R K \mathrm{d}\sigma + \sum_{j=1}^{p} \theta_j = 2\pi\chi(R)$$

where s denotes the arclength of C_i and the integral over C_i means the sum of integrals in every regular arc of C_i.

Gaussian curvature Let M be a manifold with dimension 2. Then T_pM is the only tangent plane at p. Then sectional curvature K becomes a real-valued function on M, called the *Gaussian curvature of M*.
rel. sectional curvature

Gauss-Kronecker curvature For an orientable hypersurface $M \subset \mathbf{R}^{n+1}_{\nu}$, the function det S is called the *Gauss-Kronecker curvature* , where the shape operator S of M derive from a unit vector field.

Gauss lemma Let $o \in M$ and $0 \neq x \in T_oM$. If $v_x, w_x \in T_x(T_o(M))$ with v_x radial, then

$$\langle \mathrm{dexp}_o(v_x), \mathrm{dexp}_o(w_x) \rangle = \langle v_x, w_x \rangle.$$

The above result is called the *Gauss lemma*.

Gauss Theorema Egregium The Gaussian curvature K of a surface is invariant by local isometries.

generalized Euclidean space Let I be an arbitrary index set and $\mathbf{R}^I$ denote the I-fold product of $\mathbf{R}$ with itself. An element of $\mathbf{R}^I$ is a function from I to $\mathbf{R}$, denoted in "tuple notation" by $(x_\alpha)_{\alpha \in I}$. The *generalized Euclidean space* E^I is the subset of $\mathbf{R}^I$ consisting of all points $(x_\alpha)_{\alpha \in I}$ such that $x_\alpha = 0$ for all but finitely many values of α. E^I is topoloized by the metric

$$|x - y| = \max_{\alpha \in I} \{|x_\alpha - y_\alpha|\}.$$

facta. If ϵ_α is the map of I into $\mathbf{R}$ whose value is 1 on the index α and 0 on all other element of I, then the set $\{\epsilon_\alpha | \alpha \in I\}$ is a basis for E^I. (Of course, it is not a basis for $\mathbf{R}^I$.)

general linear group Let K be a field. The *general linear group* $GL(n, K)$ is the multiplicative group of all nonsingular $n \times n$ matrices over K.
facta. For any $a \in GL(n, K)$, det $a \neq 0$.
rel. special linear group

genus of a surface Every compact connected Riemannian surfaces $S \subset \mathbf{R}^3$ is homeomorphic to a sphere with a certain number g of the handles. The number

$$g = \frac{2 - \chi(S)}{2}$$

is called the *genus of* S, where $\chi(S)$ is the Euler-Poincaré characteristic of S.

geodesic A *geodesic in a semi-Riemannian manifold* M is a curve $\gamma : I \longrightarrow M$ is a curve whose vector field γ' is parallel.
facta. 1. Geodesics are the curves of acceleration zero : $\gamma'' = 0$.
2. When γ is a geodesic in M, γ' is an integral curve of some vector field.
3. If α is an integral curve of a vector field, then $\pi \circ \alpha$ is a geodesic in M.

geodesically complete The spacetime is *geodesically complete* if all inextendible geodesics are complete.

geodesically connected For a manifold M, M is *geodesically connected* if arbitrary points in M can be joined by any geodesic at all, much less a minimizing one.
facta. It is equivalent to all exponential maps of M being onto.

geodesically singular spacetime A causal incomplete spacetime is called a *geodesically singular spacetime* .

geodesic flow *refer to* geodesic spray

geodesic spray Let $\mathbf{g}$ be a semi-Riemannian metric on M and let X_E be the associated Lagrangian vector field for $L(v) = \frac{1}{2}\mathbf{g}(v, v)$. X_E is called the *geodesic spray* and its flow is called the *geodesic flow*.

geodesic variation *refer to* Jacobi vector field

geometric realization If the abstract simplicial complex $\mathcal{S}$ is isomorphic with the vertex scheme of the simplicial complex K, we call K a *geometric realization of* $\mathcal{S}$. It is uniquely determined up to a linear isomorphism.
rel. vertex scheme

germ of a function Let M be a manifold. Functions f and g defined on open sets containing $m \in M$ are said to have the same *germ at* m if they agree on some neighborhood of m. This introduces an equivalence relation on the smooth function defined on neighborhoods of m, two functions being equivalent if and only if they have the same germ. The equivalence classes are called the *germs* and we denote the set of all germs at m by $\widetilde{F}_m$. *facta.* The operations of addition, scalar multiplication and multiplication of functions on $\widetilde{F}_m$ induces the structure of algebra over **R**.

global hyperbolicity A subset H of M is *globally hyperbolic* provided the strong causality condition holds and for each $p < q$ in H, the set $J(p,q) = J^+(p) \bigcap J^-(q)$ is compact and contained in H.

global isometry *refer to* symmetric space

global time function A continuous function $f : M \longrightarrow \mathbf{R}$ is a *global time function* if f is strictly increasing along each future pointing causal curve.

gradient The *gradient* grad f *of a function* $f \in \mathcal{F}(M)$ is the vector field metrically equivalent to the differential $\mathrm{d}f \in \mathfrak{X}^\star(M)$. Thus

$$\langle \mathsf{grad}\ f, X\rangle = \mathrm{d}f(X) = Xf,$$

for all $X \in \mathfrak{X}(M)$.

Grassman algebra *refer to* exterior algebra

Grassman manifold Unoriented p-subspace of n-space, where $n = p + q$.

$$G_{pq} = O(p+q)/O(p) \times O(q) = SO(p+q)/S(O(p) + O(q)),$$

where $O(n)$ is a n-dimensional orthogonal group .
Let G_{pq} be the set of all p-dimensional subspaces of $\mathbf{R}^n$, $n = p + q$, then G_{pq} is called *Grassman manifold* .

Green's identities Let M be an oriented Riemannian manifold. Let f and g be smooth functions on M and D a regular domain in M. If $\vec{n}$ is the unit outer normal vector field along ∂D and we denote $\partial g/\partial n$ by $\vec{n}(g)$, then

$$\int_{\partial D} f\frac{\partial g}{\partial n} = \int_D \langle \mathsf{grad}\ f, \mathsf{grad}\ g\rangle - \int_D f \bigtriangleup g \quad \text{(Green's first identity);}$$

$$\int_{\partial D}\left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) = \int_D (g \bigtriangleup f - f \bigtriangleup g) \quad \text{(Green's second identity),}$$

where $\bigtriangleup f$ is the Laplacian of f.

Green's operator Let $E^p(M)$ and $H^p(M)$ be sets of differential p-forms and harmonic p-forms, respectively. Define the *Green's operator* $G : E^p(M) \longrightarrow (H^p)^\perp$ by setting $G(\alpha)$ equal to the unique solution of $\bigtriangleup\omega = \alpha - H(\alpha)$ in $(H^p)^\perp$.

group of affine motion Let K be the product manifold $GL(n, \mathbf{R}) \times \mathbf{R}^n$. By setting $(A_1, v_1)(A_2, v_2) = (A_1A_2, A_1v_2 + v_1)$, we have a group structure on K and hence K becomes a Lie group. This Lie group is the *group of affine motion of* $\mathbf{R}^n$ when if we identify the element (A, v) of K with the affine motion $x \mapsto Ax + v$ of $\mathbf{R}^n$, then the operation in K is composition of affine motions.

H

Hadamard's theorem Let H be a complete, simply connected Riemannian manifold with sectional curvature $K \leq 0$. Then for each $p \in H$, the exponential map $\exp_p : T_pH \longrightarrow H$ is a diffeomorphism . In particular,

i. H is diffeomorphic to $\mathbf{R}^n$.

ii. For $p, q \in H$, there is a unique geodesic $\gamma : \mathbf{R} \longrightarrow M$ such that $\gamma(0) = p$ and $\gamma(1) = p$.

facta. Such manifolds have the same manifold structure as Euclidean space and share the well-known Euclidean geometric property : "two points determines a line".

half-density class *refer to* intrinsic Hilbert space $\mathcal{H}(Q)$

Hamiltonian of a system *refer to* mechanical system with symmetry

Hamiltonian operator If the classical potential energy is given by the function V on a manifold Q, the *Hamiltonian operator* is defined by

$$H_{op} = -\frac{1}{2} \triangle_{\mathbf{g}} + Q_V$$

on $\mathcal{H}(Q)$.

Hamiltonian system Let (M, ω) be a symplectic manifold and $H : M \longrightarrow \mathbf{R}$ a given C^r function. The vector field X_H determined by

$$\omega(X_H, Y) = \mathrm{d}H \cdot Y;$$

that is,

$$i_{X_H}\omega = \mathrm{d}H$$

is called the *Hamiltonian vector field* with energy function H. The triple (M, ω, X_H) is a *Hamiltonian system.*

Hamiltonian vector field *refer to* Hamiltonian system

harmonic p-form For the Laplacian $\triangle$, let

$$H^p = \{\omega \in E^p(M) : \triangle\omega = 0\}.$$

Then the element of H^p are called *harmonic p-forms*.

Housdorff metric Let S be a metric space with metric d and 2^S denote the set of all subsets of S. For $a \in S$ and a nonempty subset B of S, define

$$d(a, B) = \inf\{d(a, b) : b \in B\}$$

and for nonempty subsets A and B of S,

$$\vec{d}(A, B) = \sup\{d(a, B) : a \in A\}.$$

As this not symmetric, we define

$$\overline{d}(A, B) = \sup\{\vec{d}(A, B), \vec{d}(B, A)\}.$$

If $A \neq \emptyset$ and $B \neq \emptyset$, define $d(a, B) = \infty$ and $\overline{d}(A, B) = \infty$. Finally, define $\overline{d}(\emptyset, \emptyset) = 0$. The metric $\overline{d}$ is called the *Housdorff metric*.

Housdorff space A topological space S is called *Housdorff* if each two distinct points have disjoint closed sets have disjoint neighborhoods (that is, with empty intersection.)
facta. A space S is Housdorff iff the *diagonal set* $\triangle_S = \{(u, u) : u \in S\}$ of S is closed in $S \times S$ in the product topology.

Heine-Borel theorem *refer to* Hopf-Rinow theorem

helix A curve α is called a *helix* if the tangent lines of α makes a constant angle with a fixed direction.

hemisphere Let M be a semi-Riemannian manifold. The *upper hemisphere* E_+^{n-1} *of* S^{n-1} consists of all points $p = (p_1, \ldots, p_n)$ in S^{n-1} for which $p_n \geq 0$. Similarly, the *lower hemisphere* E_-^{n-1} *of* S^{n-1} consists of all points $p = (p_1, \ldots, p_n)$ in S^{n-1} for which $p_n \leq 0$.

Hermitian scalar product A *Hermitian scalar product on a complex vector space* V is a function $h : V \times V \longrightarrow \mathcal{C}$ such that

i. $h(v, w)$ is $\mathcal{C}$-linear in v;
ii. $h(w, v) = h(v, w)$;
iii. h is nondegenerate ; that is, $h(v, w) = 0$ for all w implies $v = 0$.

Hessian The *Hessian of a function* $f \in \mathcal{F}(M)$ is its second covariant differential $H^f = D(Df)$.
facta. $H^f(X, Y) = XYf - (D_X Y)f = \langle D_X(\text{grad } f), Y\rangle$.

Hodge decomposition theorem For each integer p with $0 \leq p \leq n$, H^p is finite dimensional and we have the following orthogonal directed sum decompositions of the space $E^p(M)$ of smooth p-forms on M:

$$\begin{aligned} E^p(M) &= \triangle(E^p) \oplus H^p \\ &= \mathrm{d}\delta(E^p) \oplus \delta\mathrm{d}(E^p) \oplus H^p \\ &= \mathrm{d}(E^{p-1}) \oplus \delta(E^{p+1}) \oplus H^p. \end{aligned}$$

Consequently, the equation $\triangle\omega = \alpha$ has a solution $\omega \in E^p(M)$ if and only if the p-form α is orthogonal to the space of harmonic p-forms.

Hodge star operator Let M be a Riemannian n-manifold and let ω be a k-form. Define an $(n-k)$-form $\star\omega$ by

$$(\star\omega)(v_{k+1}, \ldots, v_n) = \omega(v_1, \ldots, v_k)$$

where $v_1, \ldots, v_n$ are oriented orthonormal vectors in T_pM for fixed $p \in M$. The operator $\star$ is called *Hodge star operator*.
facta. 1. $\star\star = (-1)^{k(n-k)}$.
2. For any $v, w \in \Lambda_k(V)$, the inner product is

$$\langle v, w\rangle = \star(w \wedge \star v) = \star(v \wedge \star w).$$

$\mathsf{Hom}\,(A, G)$ If A and G are abelian groups, then $\mathsf{Hom}\,(A, G)$ is the set of all homomorphisms of A into G.
facta. $\mathsf{Hom}\,(A, G)$ becomes an abelian group if we add two homomorphisms by adding their values in G.

homogeneous manifold A semi-Riemannian manifold M is *homogeneous* provided that given any points $p, q \in M$, there is an isometry ϕ of M such that $\phi(p) = q$.
facta. A symmetric semi-Riemannian manifold is homogeneous.

homologous chains Two chains c and c' are *homologous* if $c - c' = \partial_{p+1}d$ for some $(p+1)$-chain d. In particular, if $c = \partial_{p+1}d$, c is said to be *homologous to zero* or simply that c *bounds*.

homology group *refer to* chain complex

homology theory If $\mathcal{A}$ is admissible, a *homology theory on* $\mathcal{A}$ consists of three functions

i. A function H_p defined for each integer p and each pair (X, A) in $\mathcal{A}$ whose value is an abelian group .

ii. A function that for each integer p, assigns to each continuous map $h : (X, A) \longrightarrow (Y, B)$ a homomorphism

$$(h_\star)_p : H_p(X, A) \longrightarrow H_p(Y, B).$$

iii. A function that for each integer p, assigns to each pair (X, A) in $\mathcal{A}$, a homomorphism

$$(\partial_\star)_p : H_p(X, A) \longrightarrow H_{p-1}(A),$$

where A denotes the pair $(A, \emptyset)$.

These functions are to satisfy the following axioms where all pairs of spaces are in $\mathcal{A}$. As usual, we shall simplify notation and delete the dimensional subscripts on $h_\star$ and $\partial_\star$.

Axiom 1 If i is the identity map, then $i_\star$ is the identity.

Axiom 2 $(k \circ h)_\star = k_\star \circ h_\star$.

Axiom 3 If $f : (X, A) \longrightarrow (Y, B)$, then the following diagram commutes:

$$\begin{array}{ccc} H_p(X, A) & \xrightarrow{f_\star} & H_p(Y, B) \\ \downarrow \partial_\star & & \downarrow \partial_\star \\ H_{p-1}(A) & \xrightarrow{(f|A)_\star} & H_{p-1}(B) \end{array}$$

Axiom 4 (*Exactness axiom*) The sequence

$$\cdots \longleftarrow H_p(A) \xleftarrow{i_\star} H_p(X) \xleftarrow{\pi_\star} H^p(X, A) \xleftarrow{\partial_\star} H^{p-1}(A) \longleftarrow \cdots$$

is exact, where $i : A \longrightarrow X$ and $\pi : X \longrightarrow (X, A)$ are inclusion maps.

Axiom 5 (*Homotopy axiom*) If h and k are homotopic, then $h_\star = k_\star$.

Axiom 6 (*Excision axiom*) Given (X, A), let U be an open set in X such that $\overline{U} \subset \operatorname{int} A$. If $(X - U, A - U)$ is admissible, then the inclusion induces a homology isomorphism

$$H_p(X - U, A - U) \simeq H_p(X, A).$$

Axiom 7 (*Dimension axiom*) If P is an one-point space, then $H_p(P) = 0$ for $p \neq 0$ and $H_0(P) \simeq G$, where G is a fixed abelian group.

Axiom 8 (*Axiom of compact support*) If $\alpha \in H_p(X, A)$, there is an admissible pair (X_0, A_0) with X_0 and A_0 compact such that α is in the image of the homomorphism $H_p(X_0, A_0) \longrightarrow H_p(X, A)$ induced by inclusion.

Above axioms are called the *Eilenberg-Steenrod axioms*.

homothety A diffeomorphism $\psi : M \longrightarrow N$ of semi-Riemannian manifolds such that $\psi^\star(\mathbf{g}_N) = c\mathbf{g}_M$ for some constant $c \neq 0$ is called a *homothety of coefficient* c. If $c = 1$, then ψ is called *distance preserving*. If $c = -1$, we call ψ an *anti-isometry*.
rel. conformal mapping

Homotopy axiom *refer to* homology theory

homotopy equivalence Two spaces X and Y are said to be *homotopy equivalent* or to have the same *homotopy type* if there are maps

$$f : X \longrightarrow Y \text{ and } g : Y \longrightarrow X$$

such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. The maps f and g are often called *homotopy equivalences* and g is said to be *homotopy inverse of* f.

homotopy inverse *refer to* homotopy equivalence

homotopy operator in cohomology Let f_1 and f_2 be smooth maps of M into N. Then we have induced maps

$$\delta f_i : E^k(N) \longrightarrow E^k(M) \text{ for each } k.$$

In de Rham cohomology, a *homotopy operator for* f_1 *and* f_2 is a collection of linear transformations

$$h_k : E^k(N) \longrightarrow E^{k-1}(M)$$

such that

$$h_{k+1} \circ \mathrm{d} + \mathrm{d} \circ h_k = \delta f_1 - \delta f_2.$$

homotopy sphere theorem Let M be a complete simply connected Riemannian manifold such that $K \geq C > 0$ with $\mathrm{diam}(M) > \frac{\pi}{2\sqrt{C}}$ where K is the sectional curvature of M and C is a constant curvature. Then M is called a *homotopy sphere*.
cf. Poincaré conjecture

homotopy type *refer to* homotopy equivalence

Hopf-Rinow theorem For a connected Riemannian manifold M, the following conditions are equivalent:

i. As a metric space under Riemannian distance d, M is complete; that is, every Cauchy sequence converges. (*metrically completeness*)
ii. There exists a point $p \in M$ from which M is geodesically complete; that is, $\exp_p$ is defined on the entire tangent space T_pM.
iii. For any $v \in TM$, the geodesic $\gamma(t)$ in M with $\gamma'(0) = v$ is defined for all $t \in \mathbf{R}$. (*geodesic completeness*)
iv. Every closed bounded subset of M is compact. (*Heine-Borel theorem*)

Hopf's criteria A complete, simply connected, n-dimensional Riemannian manifold of constant curvature C is isometric to

$$\begin{aligned} \text{the sphere} \quad & \text{if} \quad C = 1/r^2, \\ \text{Euclidean space} \quad \mathbf{R}^n \quad & \text{if} \quad C = 0, \\ \text{hyperbolic space} \quad H^n(r) \quad & \text{if} \quad C = -1/r^2. \end{aligned}$$

Hopf trace theorem Let K be a finte complex and let $\phi : C_p(K) \longrightarrow C_p(K)$ be a chain map. Then

$$\sum_p (-1)^p \mathsf{trace}(\phi, C_p(K)) = \sum_p (-1)^p \mathsf{trace}(\phi_\star, H_p(K)/T_p(K)),$$

where $T_p(K)$ is a torsion subgroup of the homology group $H_p(K)$.

Hubble law All distant galaxies are moving away from the earth at a rate proportional to their distance from the earth.

Hubble number Let $M(k, f)$ be a Robertson-Walker spacetime . The number $H_0 = f_0'/f_0$ is called the *Hubble number*. The *Hubble time* is defined by H_0^{-1}.

Hubble time *refer to* Hubble number

hyperbolic angle If v and w be timelike vectors in the same timecone of Lorentz vector space, there is a unique number $\varphi \geq 0$, called the *hyperbolic angle between v and w*, such that

$$\langle v, w \rangle = -|v|\,|w| \cosh \varphi.$$

hyperquadric For $r > 0$ and $\varepsilon = \pm 1$, $Q = q^{-1}(\varepsilon r^2)$ is called *(central) hyperquadrics of* $\mathbf{R}_\nu^{n+1}$. It is a semi-Riemannian hypersurface of $\mathbf{R}_\nu^{n+1}$ with unit normal $U = P/r$ and sign ε.

hyperregular Hamiltonian Let M be a manifold and $H \in \mathcal{F}(T^\star M)$. Then H is called a *hyperregular Hamiltonian* if $\mathcal{F}H : T^\star M \longrightarrow T^\star M$ is a diffeomorphism.

hyperregular Lagrangian Let M be a manifold and $L \in \mathcal{F}(TM)$. Then L is called a *hyperregular Lagrangian* if $\mathcal{F}L : TM \longrightarrow T^\star M$ is a diffeomorphism.

hypersurface A *hypersurface* in a manifold M is a submanifold $\mathcal{S}$ whose codimension (dim $M-$ dim $\mathcal{S}$) is 1.

I

image of a function *refer to* kernel of a function

imbedding An *imbedding of a manifold P into M* is an one-to-one immersion $\phi : P \longrightarrow M$ such that the induced map $P \longrightarrow \phi(P)$ is a homeomorphism onto the subspace $\phi(P)$ of M. The term *embedding* is often used instead of imbedding.

immersed submanifold If the inclusion map $\jmath : P \longrightarrow M$ is an immersion, we call P an *immersed submanifold of M*.

immersion A map $\phi : M \longrightarrow N$ is an *immersion* provided that ϕ is a smooth map such that $\mathrm{d}\phi_p$ is one-to-one for all $p \in M$.

implicit function theorem Let $U \subset M$, $V \subset N$ be open and $f : U \times V \longrightarrow G$ be C^r, $r \geq 1$. For some $u_0 \in U$ and $v_0 \in V$, assume a map $D_2 f : N \longrightarrow G$ which assigns n to $Df(u_0, v_0) \cdot (n, 0)$ is an isomorphism. Then there are neighborhoods U_0 of u_0 and W_0 of $f(u_0, v_0)$ and a unique C^r map $g : U_0 \times W_0 \longrightarrow V$ such that for all $(u, w) \in U_0 \times W_0$,

$$f(u, g(u, w)) = w.$$

On Euclidean space: Let $U \subset \mathbf{R}^{c-d} \times \mathbf{R}^d$ be open and let $f : U \longrightarrow \mathbf{R}^d$ be smooth. Let $(r_1, \ldots, r_{c-d}, s_1, \ldots, s_d)$ be the canonical coordinate system on $\mathbf{R}^{c-d} \times \mathbf{R}^d$. Suppose that at the point $(r_0, s_0) \in U$

$$f(r_0, s_0) = 0,$$

and that the matrix

$$\left\{ \left. \frac{\partial f_i}{\partial s_j} \right|_{(r_0, s_0)} \right\}_{1 \leq i, j \leq d}$$

is nonsingular. Then there exists an open neighborhood V of r_0 in $\mathbf{R}^{c-d}$ and an open neighborhood W of s_0 in $\mathbf{R}^d$ such that $V \times W \subset U$ and there exists a smooth map $g : V \longrightarrow W$ such that for each $(p, q) \in V \times W$,

$$f(p, q) = 0 \ \text{ iff } \ q = g(p).$$

incompressible vector field A vector field X on a manifold M is *incompressible* provided $\mathsf{div} X = 0$.

indecomposable future set An *indecomposable future set IF* is an open future set that cannot be written as a union of two proper subsets both of which are open future sets.

indecomposable past set An *indecomposable past set IP* is an open past set that cannot be written as a union of two proper subsets both of which are open past sets.

index The *index ν of a symmetric bilinear form b on V* is the largest integer that is the dimension of spaces $W \subset V$ on which $b|W$ is negative definite.

index form The *index form I_σ of a nonnull geodesic $\sigma \in \Omega(p,q)$* is the unique symmetric bilinear form

$$I_\sigma : T_\sigma(\Omega) \times T_\sigma(\Omega) \longrightarrow \mathbf{R}$$

such that if $V \in T_\sigma(\Omega)$, then

$$I_\sigma(V,V) = L_p''(0),$$

where p is any fixed endpoint variation of σ with variation vector field V.

induced connection If M is a semi-Riemannian submanifold of $\overline{M}$, the Levi-Civita connection $\overline{D}$ of $\overline{M}$ gives rise in a natural way to a function $\mathfrak{X}(M) \times \overline{\mathfrak{X}}(M) \longrightarrow \overline{\mathfrak{X}}(M)$ called the *induced connection on $M \subset \overline{M}$*.

inextendible manifold *refer to* extendible manifold

infinitesimally symplectic mapping A linear mapping $u : E \longrightarrow E$ is *infinitesimally symplectic* with respect to a symplectic form ω if $\omega(ue, e') + \omega(e, ue') = 0$ for all $e, e' \in E$; that is, if u is ω-skew. Let $\mathrm{sp}(E, \omega)$ denote the set of all linear mappings from E to E that are infinitesimally symplectic with respect to ω.

inner ball In arbitrary spacetimes, the *future inner ball $B^+(p,\epsilon)$* [*past inner ball $B^-(p,\epsilon)$*] of $I^+(p)$ [$I^-(p)$] is given by

$$B^+(p,\epsilon) = \{q \in I^+(p) : d(p,q) < \epsilon\}$$

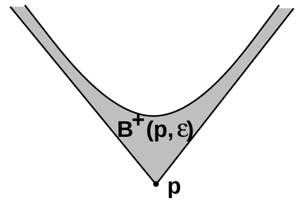
and

$$B^-(p,\epsilon) = \{q \in I^-(p) : d(q,p) < \epsilon\},$$

respectively.
facta. Inner balls need not be open.

inner continuous Let $(M, \mathbf{g})$ be a spacetime . The set-valued function I^+ is said to be *inner continuous at $p \in M$* if for each compact set $K \subset I^+(p)$ there exists a neighborhood $U(p)$ of p such that $K \subset I^+(q)$ for each $q \in U(p)$. Inner continuity of I^- may be defined dually.
cf. outer continuous

inner product An *inner product* is a positive definite scalar product.
facta. The canonical example of inner product is a dot product on $\mathbf{R}^n$, for which $v \cdot w = \sum v_i w_i$.



instantaneous observer Let M be a Minkowski spacetime . A timelike future pointing unit vector $u \in T_pM$ is called an *instantaneous observer at* p.

integral curve A curve $\alpha : I \longrightarrow M$ is an *integral curve of* $V \in \mathfrak{X}(M)$ provided that $\alpha' = V_\alpha$; that is, $\alpha'(t) = V_{\alpha(t)}$ for all $t \in I$.

intergal manifold of a distribution A submanifold (N, ψ) of a manifold M is an *integral manifold of a distribution* $\mathcal{D}$ *on* M if

$$d\psi(T_nN) = \mathcal{D}(\psi(m)) \text{ for each } n \in N.$$

facta. Let $\mathcal{D}$ be a smooth distribution on M such that through each point of M, there passes an integral manifold of $\mathcal{D}$. Then $\mathcal{D}$ is involutive.

interior of a manifold *refer to* boundary of a manifold

interior of a set Let S be a topological space and $A \subset S$. Then the *interior of* A, denoted int A, is the union of all open sets contained in A.
cf. closure of a set
facta. If A is open, int $A = A$.

intrinsic Hilbert space $\mathcal{H}(Q)$ Let Q be a manifold. Consider the set of all pairs (f, μ), where μ is a natural measure and f is a complex measurable function such that

$$\int_Q |f|^2 d\mu < \infty.$$

Two pairs (f, μ) and (g, ν) will be called *equivalent* provided that $f\sqrt{d\mu/d\nu} = g$. The equivalence class of (f, ν) is denoted by $f\sqrt{d\mu}$. $\mathcal{H}(Q)$ is the set of all such equivalence classes. The Hilbert space structure of $\mathcal{H}(Q)$ can be defined as follows:
Pick any natural measure μ. Then the map U_μ which assigns f to $f\sqrt{d\mu}$ is a bijection from $L^2(Q, \mu)$ onto $\mathcal{H}(Q)$. By use of U_μ, transfer the Hilbert space structure from $L^2(Q, \mu)$ to $\mathcal{H}(Q)$. Such a space $\mathcal{H}(Q)$ is called *intrinsic Hilbert space of* Q. Every member of $\mathcal{H}$ is called *half-density*.

invariance of domain Let U be open in $\mathbf{R}^n$ and let $f : U \longrightarrow \mathbf{R}^n$ be continuous and injective. Then $f(U)$ is open in $\mathbf{R}^n$ and f is an imbedding. Such property is called the *"invariance of domain"*.

invariant k-form Let M be a manifold and X a vector field on M. Let $\omega \in \Lambda^k(M)$. Then ω is an *invariant k-form of X* if $L_X \omega = 0$.
cf. Killing vector field

invariant manifold If S is a submanifold of M and $X \in \mathfrak{X}(M)$, then S is an *invariant manifold of X* if for all $p \in S$, $X(p) \in T_pS \subset T_pM$.

invariant metric tensor If a Lie group G acts on a manifold M, a metric tensor on M is *G-invariant* provided that for each $g \in G$, the diffeomorphism $p \mapsto gp$ is an isometry.

inverse function theorem Let $\phi : M \longrightarrow N$ be a smooth mapping. The differential map $d\phi_p$ at a point $p \in M$ is a linear isomorphism if and only if there is a neighborhood U of p in M such that $\phi|_U$ is a diffeomorphism from U onto a neighborhood $\phi(U)$ of $\phi(p)$ in N.

involution Let (P, ω) be a symplectic manifold, $H \in \mathcal{F}(P)$ a Hamiltonian and $f_1 = H$, $f_2, \ldots, f_k$ constants of the motion (i.e., $\{f_i, H\} = 0$ for all $1 \le i \le k$). The set $\{f_1, \ldots, f_k\}$ is said to be *involution* if $\{f_i, f_j\} = 0$ for all $1 \le i, j \le k$.

involutive distribution A smooth distribution $\mathcal{D}$ is called *involutive* (or *completely integrable*) if $[X, Y] \in \mathcal{D}$ whenever X and Y are smooth vector fields lying in $\mathcal{D}$.

involutive map For a map ξ of a manifold M, ξ is *involutive* if $\xi^2 = \mathrm{id}$.

isolated element *refer to* discrete topology

isometric imbedding *refer to* isometric immersion

isometric immersion Let M and $\overline{M}$ be semi-Riemannian manifolds with metric tensors $\mathbf{g}$ and $\overline{\mathbf{g}}$, respectively. An *isometric immersion of M into $\overline{M}$* is a smooth immersion such that $\phi^\star(\overline{\mathbf{g}}) = \mathbf{g}$. An *isometric imbedding* is a one-to-one isometric immersion.

isometry Let M and N be semi-Riemannian manifolds with metric tensors $\mathbf{g}_M$ and $\mathbf{g}_N$. An *isometry from M to N* is an diffeo $\phi : M \longrightarrow N$ that preserves metric tensors $\phi^\star(\mathbf{g}_N) = \mathbf{g}_M$.

isotropic Let $I(M)$ be the set of all isometries $M \to M$. A manifold M is said to be *isotropic at $p \in M$* provided that if $v, w \in T_pM$ have $\langle v, v\rangle = \langle w, w\rangle$, there is an isometry $\phi \in I(M)$ such that $d\phi(v) = w$.

isotropy group Given $p \in M$, the *isotropy group $I_p(M)$ of M at p* is the closed subgroup $I_p(M) = \{\phi \in I(M) : \phi(p) = p\}$ of $I(M)$ consisting of all isometries of M which fix p.
facta. Given any $\phi \in I_p(M)$, the differential of ϕ maps T_pM onto T_pM since $\phi(p) = p$. *rel.* action of a Lie group

isotropy subgroup *refer to* action of a Lie group

J

Jacobian function Let M and N be semi-Riemannian manifolds of the same dimension n, oriented by volume elements $\mathrm{d}M$ and $\mathrm{d}N$. If $\phi : M \longrightarrow N$ is a smooth mapping, the function $J \in \mathcal{F}(M)$ such that $\phi^\star(\mathrm{d}N) = J\mathrm{d}M$ is the *Jacobian function of* ϕ.

Jacobian matrix Let $\phi : M^m \longrightarrow N^n$ be a smooth mapping. If ξ is a coordinate system at p in M, and η is a coordinate system at $\phi(p)$ in M, then the matrix of $\mathrm{d}\phi_p$ with respect to these coordinate bases is

$$\left(\frac{\partial(y^i \circ \phi)}{\partial x^j}(p)\right)_{1\leq i\leq n, 1\leq j\leq m}$$

called the *Jacobian matrix of* ϕ *at* p *relative to* ξ *and* η.

Jacobi equation *refer to* Jacobi vector field

Jacobi identity *refer to* Lie bracket

Jacobi metric Let $\mathbf{g}$ be a semi-Riemannian metric on M and $V : M \longrightarrow \mathbf{R}$ be bounded above (if it is not, confine attention to a compact subset of M). Let $e > V(p)$ for $p \in M$. The *Jacobi metric* is defined by

$$\mathbf{g}_e = (e - V)\mathbf{g}.$$

Jacobi vector field If γ is a geodesic, a vector field Y on γ that satisfies the *Jacobi equation* $Y'' = \mathcal{R}_{Y\gamma'}\gamma'$ is called a *Jacobi vector field*. The Jacobi equation is also called the equation of *geodesic variation* .

K

Kähler manifold A semi-Riemannian manifold M with almost complex structure J is the *Kähler manifold* provided

i. J preserves the metric; that is, $\langle JX, JY\rangle = \langle X, Y\rangle$ for all $X, Y \in \mathfrak{X}(M)$;

ii. J is parallel; that is, $D_X(JY) = J(D_X Y)$ for all $X, Y \in \mathfrak{X}(M)$.

kernel of a function Let $f : G \longrightarrow H$ be a homomorphism. The *kernel of* f is the subgroup $f^{-1}(0)$ of G, the *image of* F is the subgroup $f(G)$ of H, and the *cokernel of* f is the quotient group $H/f(G)$. These groups are denoted by ker$f$, im$f$ and cok$f$, respectively.

Kepler's second law Let α be a particle of mass $m \ll \mathsf{M}$ in $\mathbf{R}^3$ and let L be the angular momentum of α per unit mass. Then the polar computation of L gives

$$r^2\varphi = L, \quad \textit{(Kepler's second law)}$$

where $|\alpha| = r$.
rel. angular momentum

Killing form The *Killing form of Lie algebra* g is the function $B : \mathrm{g} \times \mathrm{g} \longrightarrow \mathbf{R}$ given by $B(X, y) = \mathsf{trace}(\mathrm{ad}_X, \mathrm{ad}_Y)$ where $\mathrm{ad}_X : \mathrm{g} \longrightarrow \mathrm{g}$ is the mapping sending each Y to $[X, Y]$.
facta. 1. ad_X is a linear operator and by Jacobi identity, it is a Lie derivation.
2. The Killing form B of g is a symmetric bilinear form that is invariant under all automorphisms of g and satisfies $B([X, Y], Z) = B(X, [Y, Z])$ for $X, Y, Z \in \mathrm{g}$.

Killing vector field A *Killing vector field on a semi-Riemannian manifold* is a vector field X for which the Lie derivative of the metric tensor vanishes: $L_X\mathbf{g} = 0$.

kinetic energy of a system *refer to* mechanical system with symmetry

Kobayashi's proposition A homogeneous Riemannian manifold with $K \leq 0$ and $\mathsf{Ric} < 0$ is simply connected . The result follows from these three facts about a homogeneous Riemannian manifold M:

i. Every maximal geodesic of M is either one-to-one or periodic.

ii. If M is not simply connected, it contains a periodic geodesic.

iii. If $K \leq 0$ and $\mathsf{Ric} < 0$, then M contains no periodic geodesics.

Koszul formula When D be the Levi-Civita connection , the *Koszul formula* is

$$\begin{aligned} 2\langle D_V W, X\rangle &= V\langle W, X\rangle + W\langle X, V\rangle - X\langle V, W\rangle \\ &\quad -\langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle. \end{aligned}$$

rel. Levi-Civita connection

Kronrcker delta The *Kronecker delta* δ_{ij} is a real-valued function defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Sometimes it is denoted by δ^i_j and is often called *Kronecker index*.

Kronrcker index If $\mathcal{C} = \{C_i, \partial\}$ is a chain complex, there is a map

$$\mathsf{Hom}\ (C_p, G) \times C_p \longrightarrow G$$

which carries the pair (c^p, c_p) to the element $\langle c^p, c_p\rangle$ of G. It is bilinear and is called the "evaluation map". It induces a bilinear map

$$H^p(\mathcal{C}; G) \times H_p(\mathcal{C}) \longrightarrow G$$

which is called the *Kronecker index*.
cf. Kronecker delta

Kronecker map The *Kronecker map*

$$\kappa : H^p(\mathcal{C}; G) \longrightarrow \mathsf{Hom}\ (H_p(\mathcal{C}), G)$$

sends α to the homomorphism $\langle \alpha, \cdot\rangle$. Formally, we define the map κ as

$$(\kappa\alpha^p)(\beta_p) = \langle \alpha^p, \beta_p\rangle,$$

using the Kronecker index.
facta. The map κ is a homomorphism.

Kruskal plane Let Q be the region in the uv-plane given by $uv > -\frac{2\mathsf{M}}{3}$ with line element

$$\mathrm{d}s^2 = 2F(r)\mathrm{d}u\mathrm{d}v,$$

where $F(r) = (s\mathsf{M}^2/r)\mathrm{e}^{1-r/2\mathsf{M}}$. Then it is called the *Kruskal plane of mass* M.
rel. Schwarzschild spacetime

Kruskal spacetime Let Q be a Kruskal plane of mass M, and let S^2 be the unit 2-sphere. The *Kruskal spacetime of mass* M is the warped product $K = Q \times_r S^2$, where r is the function on Q characterized by $f(r) = uv$.

And the region $v > 0$ in Kruskal spacetime is called a *truncated Kruskal spacetime* .
facta. 1. K is the smooth manifold $Q \times S^2$ furnished with line element $2F(r)\mathrm{d}u\mathrm{d}v + r^2\mathrm{d}\sigma^2$.
2. Kruskal spacetime is Ricci flat but not flat.
3. The *timefunction* $t = 2\mathrm{M}\ln|v/u|$ is defined on the open quadrants of Q.
rel. Schwarzschild spacetime

Kulkarni's theorem Let p be a point of a semi-Riemannian manifold of indefinite metric. Let $\mathcal{N}$ be a well-defined function from the set of degenerate planes in T_pM to the set $\{-1, 0, 1\}$ by $\mathrm{sgn}\langle R_{XY}X, Y\rangle$. The following conditions on T_pM are equivalent:

i. K is constant,
ii. $\mathcal{N} = 0$,
iii. $a \leq K$ or $K \leq b$, where $a, b \in \mathbf{R}$,
iv. $a \leq K \leq b$ on indefinite planes,
v. $a \leq K \leq b$ on definite planes.

Künneth theorem for cohomology Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes that vanish below a certain dimension. Suppose $\mathcal{C}$ is free and finitely generated in each dimension. Then there is a natural exact sequence

$$\begin{aligned} 0 \longrightarrow \oplus_{p+q=m} H^p(\mathcal{C}) \otimes H^q(\mathcal{C}') \longrightarrow H^m(\mathcal{C} \otimes \mathcal{C}') \\ \longrightarrow \oplus_{p+q=m} H^{p+1}(\mathcal{C}) * H^q(\mathcal{C}') \longrightarrow 0. \end{aligned}$$

It splits (but not naturally) if $\mathcal{C}'$ is free and finitely generated ineach dimension.

Künneth theorem for homology Let $\mathcal{C}$ be a free chain complex and let $\mathcal{C}'$ be a chain complex. There is an exact sequence

$$\begin{aligned} 0 \longrightarrow \oplus_{p+q=m} H_p(\mathcal{C}) \otimes H_q(\mathcal{C}') \longrightarrow H_m(\mathcal{C} \otimes \mathcal{C}') \\ \longrightarrow \oplus_{p+q=m} H_{p-1}(\mathcal{C}) * H_q(\mathcal{C}') \longrightarrow 0 \end{aligned}$$

which is natural with respect to homomorphisms induced by chain maps. If the cycles of $\mathcal{C}'$ are a direct summand in the chains, the sequence splits, but not naturally.

L

labelling of vertices Given a finite complex L, a *labelling of vertices of* L is an onto map f mapping the vertex set of L to a set (called the *set of labels*).

Lagrange bracket If (M, ω) is a symplectic manifold and $X, Y \in \mathfrak{X}(M)$, the *Lagrange bracket of the vector field* X *and* Y is the scalar function

$$[\![X, Y]\!] = \omega(X, Y).$$

If (U, φ) is a chart on M, the *Lagrange bracket of* φ is the matrix of functions on U given by

$$[\![u^i, u^j]\!] = \left[\!\!\left[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right]\!\!\right],$$

where $\partial/\partial u^i$ are the standard basis vectors associated with the chart (U, φ) regarded as local vector fields on M.
facta. 1. $[\![X_f, X_g]\!] = \{f, g\}$.
2. A diffeomorphism f is symplectic iff it preserves all Lagrange brackets; that is, $[\![f^\star X, f^\star Y]\!] = f^\star [\![X, Y]\!]$.

Lagrange Multiplier theorem Let $T : E \longrightarrow \mathbf{R}$ and $S : E \longrightarrow F$ be linear maps, where S is surjective and E, F are finite-dimensional vector spaces. Then T is surjective on ker$S$ if and only if $T \times S : E \longrightarrow \mathbf{R} \times F$ is surjective.

Lagrange submanifold A submanifold $L \subset P$ is called *Lagrangian* if it is isotropic and there is an isotropic subbundle $E \subset TP|_L$ such that

$$TP|_L = TL \oplus E.$$

Lagrange two-form Let ω be the canonical form on $T^\star M$ and let $L \in \mathcal{F}(TM)$. The form

$$\omega_L = (\mathcal{F}L)^\star \omega$$

is called the *Lagrange two-form*.

Laplace-Beltrami operator The *Laplace-Beltrami operator* $\bigtriangledown^2 f$ *of a function* $f \in \mathcal{F}(M)$ is the divergence of its gradient :

$$\bigtriangledown^2 f = \mathsf{div}(\mathsf{grad}\ f) \in \mathcal{F}(M).$$

Sometimes it is called the *Laplacian* and is denoted by $\triangle f$.
facta. The Laplacian of f is the contraction of its Hessian.
cf. Laplace-de Rham operator

Laplace-de Rham operator The *Laplace-de Rham operator* is defined by

$$\triangle = \mathrm{d}\delta + \delta\mathrm{d},$$

where δ is a codifferential operator. Sometimes it is called the *Laplacian*.
facta. 1. The operator $\triangle$ is symmetric and nonnegative.
2. On functions, $\triangle$ differs in sign from the Laplace-Beltrami operator $\bigtriangledown^2$.
cf. Laplace-Beltrami operator

Laplacian *refer to* 1. Laplace-Beltrami operator
2. Laplace-de Rham operator

leaf *refer to* warped product

Lefschetz fixed-point theorem Let K be a finite complex and let $h : |K| \longrightarrow |K|$ be a continuous map. If the Lefschetz number $\Lambda(h)$ is nonzero, then h has a fixed point.

Lefschetz number Let K be a finite complex and let $h : |K| \longrightarrow |K|$ be a continuous map. The number

$$\Lambda(h) = \sum_p (-1)^p \mathsf{trace}(h_\star, H_p(K)/T_p(K))$$

is called the *Lefschetz number of* h, where $T_p(K)$ is the torsion subgroup of homology group $H_p(K)$.
facta. $\Lambda(h)$ depends only on the homology class of h.

left-invariant of metric tensor A Lie group G is a group which is also an analytic manifold such that the mapping $(\mathbf{g}, \mathbf{h}) \rightarrow \mathbf{g}\mathbf{h}^{-1}$ from $G \times G \longrightarrow G$ is analytic. This multiplication induces left and right translation maps $L_\mathbf{g}, R_\mathbf{g}$, so-called *left-multiplication* and *right-multiplication*, for each $\mathbf{g} \in G$ given respectively by

$$L_\mathbf{g}(\mathbf{h}) = \mathbf{g}\mathbf{h} \text{ and } R_\mathbf{g}(\mathbf{h}) = \mathbf{h}\mathbf{g}.$$

Then a Riemannian or Lorentzian metric g for G is said to be *left-invariant* [*right-invariant*] if $\langle L_{\mathbf{g}_\star} v, L_{\mathbf{g}_\star} w\rangle = \langle v, w\rangle$ [$\langle R_{\mathbf{g}_\star} v, R_{\mathbf{g}_\star} w\rangle = \langle v, w\rangle$] for all $\mathbf{g} \in G$, $v, w \in TG$. A metric which is both left- and right-invariant is said to be *bi-invariant*.

left-multiplication *refer to* left-invariant

lens space Let n and k be relatively prime positive integers. The *lens space* $L(n, k)$ is a quotient space of the ball B^3. Its construction is as follows: The general point of B^3 in the form (z, t), where z is complex, t is real, and $|Z|^2 + t^2 \leq 1$. Let $\lambda = \exp(2\pi i/n)$. Define $f : S^2 \longrightarrow S^2$ by

$$f(x) = (\lambda^k z, -t).$$

Let's identify each point $x = (z, t)$ of the lower hemisphere E^2 of $S^2 = \mathrm{bd}B^3$ with the point $f(x)$ of upper hemisphere E^2_+. The resulting quotient space is called the *lens space* $L(n, k)$.

Levi-Civita connection On a semi-Riemannian manifold M, there is a unique connection D such that

i. $[V, W] = D_V W - D_W V$
ii. $X\langle V, W\rangle = \langle D_X V, W\rangle + \langle V, D_X W\rangle$

for all $X, V, W \in \mathfrak{X}(M)$. D is called the *Levi-Civita connection of* M and it is characterized by the Koszul formula.

Levi-Civita covariant derivative *refer to* covariant derivative

Lie algebra A *Lie algebra over* **R** is a real vector space g furnished with bilinear function [,] : g × g⟶g, called its *bracket operation*, such that for all $X, Y, Z \in$ g,

i. $[X, Y] = -[Y, X]$, (*skew-symmetric*)
ii. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. (*Jacobi identity*)

facta. gl$(n, \mathbf{R})$ should always be made a Lie algebra by defining $[x, y] = xy - yx$, where xy is matrix multiplication.

Lie bracket For $V, W \in \mathfrak{X}(M)$, let

$$[V, W] = VW - WV.$$

Then [,] is called *Lie bracket*.
facta. 1. Lie Bracket satisfies the following properties:

i. **R**-linearity :

$$\begin{aligned} [aV + bW, X] &= a[V, X] + b[W, X] \\ [X, aV + bW] &= a[X, V] + b[X, W] \end{aligned}$$

ii. skew-symmetry : $[V, W] = -[W, V]$
iii. Jacobi identity : $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Sometimes the Jacobi identity is called the *first Bianchi identity*.
2. Lie bracket is not $\mathcal{F}(M)$-bilinear.

Lie derivative For $V \in \mathfrak{X}(M)$, the tensor derivation L_V such that

$$\begin{aligned} L_V(f) &= Vf \quad \text{for all} \quad f \in \mathcal{T}(M) \\ L_V(X) &= [V, X] \quad \text{for all} \quad X \in \mathfrak{X}(M) \end{aligned}$$

is called the *Lie derivative relative to* V.

Lie exponential map Let g be the Lie algebra of G. The *Lie exponential map* $\exp : \mathrm{g} \longrightarrow G$ sends X to $\alpha_X(1)$, where α_X is the one-parameter subgroup of $X \in \mathrm{g}$.

Lie group A *Lie group* G is a smooth manifold that is also a group with smooth group operation; that is, the maps

$$\mu : G \times G \longrightarrow G \ \text{ sending } (a, b) \text{ to } ab$$

and

$$\zeta : G \longrightarrow G \ \text{ sending } a \text{ to } a^{-1}$$

are both smooth.

Lie subgroup A Lie group H is a *Lie subgroup of a Lie group* G provided H is both an abstract subgroup and an immersed submanifold of G.

Lie transformation group *refer to* action of a Lie group

lift If $f \in \mathcal{F}(M)$, the *lift of* f to $M \times N$ is

$$\tilde{f} = f \circ \pi \in \mathcal{F}(M \times N).$$

If $x \in T_pM$ and $q \in N$, then the *lift* $\tilde{x}$ of x to (p, q) is the unique vector in $T_{(p,q)}M$ such that $\mathrm{d}\pi(\tilde{x}) = x$.
If $X \in \mathfrak{X}(M)$ and $q \in N$, then the *lift* $\widetilde{X}$ of X is the vector field whose value at each point (p, q) is the lift of X_p to (p, q).
facta. The lift of $X \in \mathfrak{X}(M)$ to $M \times N$ is the unique element of $\mathfrak{X}(M \times N)$ that is π-related to X and σ-related to the zero vector field on N.

lightcone *refer to* nullcone

lightlike curve *refer to* null curve

lightlike geodesically complete A semi-Riemannian manifold M is said to be *lightlike geodesically complete* if all lightlike inextendible geodesics are complete. Sometimes a lightlike geodesically completeness is called a *null geodesically completeness*.

lightlike geodesically incomplete A semi-Riemannian manifold M is said to be *lightlike geodesically incomplete* if some lightlike geodesic is incomplete. Sometimes a lightlike geodesically incompleteness is called a *null geodesically incompleteness*.

lightlike particle A *lightlike particle* is a future poiting null geodesic $\gamma : I \longrightarrow M$, i.e., $\langle \gamma', \gamma' \rangle = 0$.

lightlike tangent vector *refer to* null tangent vector

limit sequence Let $\{\alpha_n\}$ be an infinite sequence of future pointing causal curves in M and let $\mathcal{K}$ be a convex covering of M. A *limit sequence for* $\{\alpha_n\}$ *relative to* $\mathcal{K}$ is a (finite or infinite) sequence $p = p_0 < p_1 < \cdots$ in M such that

i. For each p_i, there is a subsequence $\{\alpha_m\}$ and for each m, numbers $s_{m0} < s_{m1} < \cdots < s_{mi}$ such that
   (a) $\lim_{m\to\infty} \alpha_m(s_{mj}) = p_j$ for each $j \leq i$.
   (b) For each $j < i$, the segment $\alpha_m|[s_{mj}, s_{m(j+1)}]$ for all m, and the points p_j, p_{j+1} are contained in a single set $\mathcal{C}_i \in \mathcal{K}$.

ii. If $\{p_i\}$ is infinite, it is nonconvergent. If $\{p_i\}$ finite, it has more than one point and no strictly longer sequence satisfies (i).

linear differential operator A *linear differential operator* L *of order* l on the $\mathbf{C}^m$-valued smooth functions on $\mathbf{R}^n$ consists of an $m \times m$ matrix (L_{ij}) in which

$$L_{ij} = \sum_{[\alpha]=0}^{l} a_{ij}^{\alpha} D^{\alpha}$$

where the a_{ij}^{α} are smooth complex-valued functions on $\mathbf{R}^n$ with at least one $a_{ij}^{\alpha} \not\equiv 0$ for some i, j and for some α with $[\alpha] = l$. A differential operator L is a *periodic differential operator* if L is a linear differential operator for which a_{ij}^{α} are periodic functions.

link of a simplex *refer to* star of a simplex

link of a vertex *refer to* star of a vertex

local diffeomorphism By the property of inverse function theorem, a smooth map $\phi : M \longrightarrow N$ such that every $\mathrm{d}\phi_p$ is a linear isomorphism is called a *local diffeomorphism* .

local distance function A *local distance function* (d, U) *on a spacetime* $(M, \mathbf{g})$ is a convex normal neighborhood U together with the distance function $d : U \times U \longrightarrow \mathbf{R}$ induced on U by the spacetime $(U, \mathbf{g}|_U)$.
facta. If $p, q \in U$, then $d(p, q) = 0$ if there is no future pointing timelike geodesic segment in U from p to q. Otherwise, $d(p, q)$ is the Lorentzian arclength of the unique future pointing timelike geodesic segment in U from p to q.

local extension A *local extension of a manifold* M is a connected open subset U of M having noncompact closure in M and an extension $\widetilde{U}$ of U such that the image of U has compact closure in $\widetilde{U}$.

local isometry A smooth map $\phi : M \longrightarrow N$ of semi-Riemannian manifold is a *local isometry* provided each differential map $\mathrm{d}\phi : T_pM \longrightarrow T_{\phi(p)}N$ is a linear isometry.

locally compact space A space is called *locally compact* if each point has a neighborhood whose closure is compact.

locally connected space *refer to* connected space

locally finite A collection $\mathcal{L}$ of subsets of a space S is *locally finite* provided each point of S has a neighborhood that meets only finitely many elements of $\mathcal{L}$.
rel. fine C^r topologies

locally finite simplicial complex A simplicial complex K is said to be *locally finite* if each vertex of K belong only to finitely many simplices of K.

locally Hamiltonian vector field A vector field X on a symplectic manifold (M, ω) is called *locally Hamiltonian* if for every $p \in M$, there is a neighborhood U of p such that X restricted to U is Hamiltonian.

long exact sequence A *long exact sequence* is an exact sequence whose index set is the set of integers. That is, it is a sequence that is infinite in both directions. It may, however, begin or end with infinite string of trivial groups.

longitudinal *refer to* variation

loop Let Z be a topological space and $\gamma : [0,1] \longrightarrow Z$ a continuous map such that $\gamma(0) = \gamma(1) = p \in Z$. Such γ is called a *loop* in Z based at p. The loop γ is called *contractible* if there is a continuous map $H : [0,1] \times [0,1] \longrightarrow Z$ such that $H(t,0) = \gamma(t)$ and $H(t,1) = p$ for all $t \in [0,1]$.

$\mathsf{Lor}(M)$ Let $\mathsf{Lor}(M)$ denote the space of all Lorentz metrics for a given manifold M.

Lorentz coordinate system A *Lorentz (inertial) coordinate system in a Minkowski spacetime M* is a time-orientation-preserving isometry $\zeta : M \longrightarrow \mathbf{R}^4_1$.

Lorentzian distance function Let γ be a piecewise smooth causal curve in a Lorentz manifold M. The *Lorentzian distance function d* is defined by

$$d(p,q) = \begin{cases} \sup_\gamma L(\gamma) & \text{if } p \leq q \\ 0 & \text{otherwise} \end{cases}$$

facta. 1. It is not symmetric since if $p < q$, then $d(p,q)$ is defined but $d(q,p)$ is not defined.
2. It is not finite-valued when given manifold M is totally vicious; that is, we may have $d(p,q) = \infty$, for some $p \leq q$.
rel. arclength

Lorentz manifold The semi-Riemannian manifold with index $\nu = 1$ and dimension $n \geq 2$ is called a *Lorentz manifold* .

Lorentz vector space A scalar product space of index 1 and dimension ≥ 2 is called *Lorentz vector space* .

lower hemisphere *refer to* hemisphere

L^2 **inner product** *refer to* L^2 norm

L^2 **norm** On n-dimensional Euclidean space $\mathbf{R}^n$, the *ordinary L^2 norm of ψ over open cube* $Q = \{p \in \mathbf{R}^n : 0 < x_i(p) < 2\pi, i = 1, \ldots, n\}$ is

$$\|\psi\| = \frac{1}{(2\pi)^{n/2}} \left(\int_Q \psi \cdot \psi \right)^{\frac{1}{2}},$$

and $\langle \psi, \varphi \rangle$ is the L^2 *inner product* defined by

$$\langle \psi, \varphi \rangle = \frac{1}{(2\pi)^n} \int_Q \psi \cdot \varphi.$$

The norm $\|\psi\|_\infty$ shall denote the *uniform norm of ψ*,

$$\|\psi\|_\infty = \sup_Q |\psi|.$$

Lyapunov stable point *refer to* stable point

M

manifold *refer to* smooth manifold

manifold with boundary M is an *n-dimensional manifold with boundary* if each point has a neighborhood homeomorphic with an open set of n-dimensional Euclidean half-space H^n.

matched covering A *matched covering* $(U^\star, \sim)$ *of a smooth manifold* M is a covering $U^\star = \{U_a : a \in A\}$ of M by open sets U_a together with a relation $\sim$ on the index set A such that for all $a, b, c \in A$,

i. $a \sim a$,

ii. If $a \sim b$, then $b \sim a$,

iii. If $a \sim b$, $b \sim c$ and $U_a \cap U_b \cap U_c \neq \phi$, then $a \sim c$.

Then $\sim$ is called *matching relation* .

matching relation *refer to* matched covering

material particle A *material particle* in M is a timelike future pointing curve $\alpha : I \longrightarrow M$ such that $|\alpha'(\tau)| = 1$ for all $\tau \in I$. The parameter τ is called the *proper time* of the particle.

maximal atlas An atlas on a space S is *maximal* if it is not contained in any strictly larger atlas on S.

maximal causal curve A future pointing causal curve σ from p to q is said to be *maximal* if the length of σ is equal to metric distance from p to q; that is, $L(\sigma) = d(p, q)$.

maximal integral curve Consider the collection of all integral curves $\alpha : I_\alpha \longrightarrow M$ of V that start at $p \in M$, that is, for which $\alpha(0) = p$. Then obviously $\alpha = \beta$ on $I_\alpha \cap I_\beta$. So all these curves define a single integral curve $\alpha_p : I_p \longrightarrow M$ where $I_p = \bigcup I_\alpha$. The curve α_p is called the *maximal integral curve of V starting at p.*
facta. Every maximal integral curve is either one-to-one, simply periodic or constant.

maximal integral manifold A *maximal integral manifold* (N, ψ) *of a distribution $\mathcal{D}$ on a manifold M* is connected integral manifold of $\mathcal{D}$ whose image in M is not a proper subset of any other connected integral manifold of $\mathcal{D}$. That is, there does not exists a continuous integral manifold (N', ψ') of $\mathcal{D}$ such that $\psi(N)$ is a connected integral manifold (N', ψ') of $\mathcal{D}$ such that $\psi(N)$ is a proper subset of $\psi'(N')$. proper subset of

maximal manifold *refer to* extendible manifold

Mayer-Vietoris sequence of complexes Let K be a complex and let K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. Let $A = K_0 \cap K_1$. Then there is an exact sequence

$$\cdots \longrightarrow H_p(A) \longrightarrow H_p(K_0) \oplus H_p(K_1) \longrightarrow H_p(K) \longrightarrow H_{p-1}(A) \longrightarrow \cdots$$

called the *Mayer-Vietoris sequence of* (K_0, K_1). There is a similar exact sequence in reduced homology if A is nonempty.

mean curvature vector field The *mean curvature vector field* H *of* $M \subset \overline{M}$ can be obtained by contracting to give a normal field on M and dividing by dimension of M. Explicitly, at $p \in M$,

$$H_p = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i II(e_i, e_i),$$

where $e_1, \ldots, e_n$ is any frame on M at p and $\varepsilon_i = \langle e_i, e_i \rangle$.

Mean Ergodic theorem Let H be a Hilbert space and $U_t : H \longrightarrow H$ a strongly continuous one-parameter unitary group (i.e., U_t is unitary for each t, is a flow on H and for each $x \in H$, the map $t \mapsto U_t(x)$ is continuous). Let the closed subspace H_0 be defined by

$$H_0 = \{x \in H : U_t(x) = x \text{ for all } t \in \mathbf{R}\}$$

and let π be the orthogonal projection onto H_0. Then for any $x \in H$,

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t U_s(x) \mathrm{d}s = \pi(x).$$

The limit is called the *time average of* x and is customarily denoted $\overline{x}$.

mechanical system with symmetry A *mechanical system with symmetry* is a quadruple (M, K, V, G), where

i. M is a Riemannian manifold with metric $\mathbf{g} = \langle\ ,\ \rangle$; M is called the *configuration space* and the cotangent bundle $T^\star M$, with its canonical symplectic structure $\omega = -\mathrm{d}\theta$, the *phase space* of the system;

ii. $K \in \mathcal{F}(T^\star M)$ is the *kinetic energy* of the system defined by

$$K(\alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle_{\tau^\star_M(\alpha)},$$

where we denote $\langle\ ,\ \rangle_p$ the metric on $T^\star_p M$ given by $\langle \alpha, \beta \rangle_p = \langle \mathbf{g}^\sharp(p)(\alpha), \mathbf{g}^\sharp(p)(\beta) \rangle_p$ for $\alpha, \beta \in T^\star_p M$ and $\mathbf{g}^\sharp : T^\star M \longrightarrow M$ is the usual isomorphism of vector bundles; $\mathbf{g}^\sharp = (\mathbf{g}^\flat)^{-1}$ and $\mathbf{g}^\flat(v_p) = \langle \cdot, v_p rangle_p$;

iii. $V \in \mathcal{F}(M)$ is the *potential energy* of the system;

iv. G is a connected Lie group acting on M by an action $\Phi : G \times M \longrightarrow M$ under which the metric is invariant (i.e., Φ is an action by isometries) and V is invariant; these conditions means that

$$K \circ \Phi_g^{T^\star} \text{ and } V \circ \Phi_g = V$$

for all $g \in G$; Such G is called the *symmetric group* of the system;

v. For $H \in \mathcal{F}(T^\star M)$,

$$H = K + V \circ \tau_M^\star$$

is the *Hamiltonian* of the system.

metric Let $\overline{\mathbf{R}}^+$ denote the nonnegative real numbers with a point $\{+\infty\}$ adjoinned and topology generated by the open intervals of the form (a, b) or $(a, +\infty]$. Let M be a set. A *metric on* M is a function $d : M \times M \longrightarrow \overline{\mathbf{R}}^+$ such that

i. $d(m_1, m_2) = 0$ iff $m_1 = m_2$;

ii. $d(m_1, m_2) = d(m_2, m_1)$;

iii. $d(m_1, m_3) \leq d(m_1, m_2) + d(m_2, m_3)$ (*triangle inequality*).

The collection of subsets of M that are unions of *s-disks* $D_\varepsilon(m)$ such that

$$D_\varepsilon(m) = \{m' \in M : d(m', m) < \varepsilon\}$$

is the *metric topology* of the metric space (M, d). A topological space S is called a *metric space* if S admits a metric on S. A *pseudometric* on a set M is a function $d : M \times M \longrightarrow \overline{\mathbf{R}}^+$ that satisfies (ii), (iii) and

$$d(m, m) = 0 \text{ for all } m.$$

A topological space admits a pseudometric is called a *pseudometric space*.

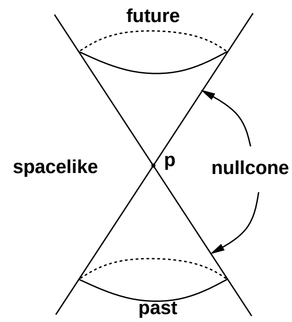
metric space *refer to* metric

metric tensor A *metric tensor* **g** *on a smooth manifold* M is a symmetric nondegenerate (0,2) tensor field on M of constant index.
facta. $\mathbf{g} \in \mathcal{T}_2^0(M)$ smoothly assigns to each point p of M a scalar product $\mathbf{g}_p$ on a tangent space T_pM, and the index of $\mathbf{g}_p$ is the same for all p.

metric topology *refer to* metric

Minkowski n-space *refer to* semi-Euclidean space

Minkowski spacetime A *Minkowski spacetime* M is a spacetime that is isometric to Minkowski 4-space $\mathbf{R}_1^4$.



Misner-completeness A semi-Riemannian manifold M is *Misner-complete* provided no geodesic races to infinity, that is, provided every geodesic $\gamma : [0, b) \longrightarrow M$, $b < \infty$, lies in a compact set.
facta. 1. If M is complete, then M is Misner-complete.
2. If M is Misner-complete, then M is inextendible.
3. A Misner-complete Riemannian manifold is complete.

momentum function Given a vector field X on M, define the associated *momentum function* $P(X) : TM \longrightarrow \mathbf{R}$ by

$$P(X)(v) = \langle X(\tau_M v), v \rangle,$$

where $\tau_M : M \longrightarrow M$ is the canonical projection. Define the *viral function* by

$$G(X) = \{E, P(X)\}.$$

momentum in Minkowski spacetime *refer to* energy in Minkowski spacetime

Morse lemma Let $f : M \longrightarrow \mathbf{R}$ be a smooth map with $p \in M$ a nondegenerate critical point; that is, $\mathrm{d}f(p) = 0$ and $D^2 f(p)$ is nondegenerate. Then there is a coordinate system about p in which p is mapped to zero and the local representative of f satisfies

$$f(x) = f(0) + \frac{1}{2} D^2 f(0) \cdot (x, x).$$

In particular, nondegenerate critical points of f are isolated.

multiplicity of a covering *refer to* covering map

Myers' theorem If M is a complete connected Riemannian manifold with $\mathsf{Ric} \geq (n-1)C > 0$ for constant C, then

i. M is compact and has diameter $\leq \pi / \sqrt{C}$.
ii. the fundamental group $\pi_1(M)$ is finite.

N

natural coordinate function For $1 \leq i \leq n$, let $u^i : \mathbf{R}^n \longrightarrow \mathbf{R}$ be the *natural coordinate function* of $\mathbf{R}^n$ if u^i sends each point $p = (p_1, \ldots, p_n)$ to its *i*-th coordinate p_i.

natural equivalence *refer to* natural transformation

natural transformation Let G and H be two functors from category C to category D. A *natural transformation T from G to H* is a rule assigning to each object X of C, a morphism

$$T_X : G(X) \longrightarrow H(X)$$

of D, such that the following diagram commutes, for all morphisms $f : X \longrightarrow Y$ of the category C:

$$\begin{array}{ccc} G(X) & \xrightarrow{T_X} & H(X) \\ \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle H(f)} \\ G(Y) & \xrightarrow{T_Y} & H(Y). \end{array}$$

If for each X, the morphism T_X is an equivalence in the category D, then T is called a *natural equivalence* of functors.

negative definite form *refer to* definite form

negative semidefinite form *refer to* definite form

nerve Let $\mathcal{A}$ be a collection of subsets of the space X. The *nerve of $\mathcal{A}$*, denoted by $N(\mathcal{A})$, is an abstract simplicial complex whose vertices are the elements of $\mathcal{A}$ and whose simplices are the finite subcollections $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ such that

$$A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset.$$

Newtonian force If $\alpha : I \longrightarrow E$ is a Newtonian particle of mass m, then the *force* on α is the vector field $\frac{\mathrm{d}m\alpha'}{\mathrm{d}t}$ on α.

Newtonian kinetic energy If $\alpha : I \longrightarrow E$ is a Newtonian particle of mass m, then the *kinetic energy* of α is the function $mv^2/2$ on I, where $v = |\alpha'|$.

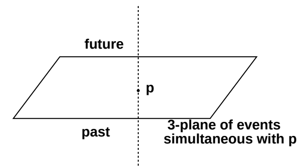
Newtonian momentum If $\alpha : I \longrightarrow E$ is a Newtonian particle of mass m, then the *momentum* of α is the vector field $m\alpha'$ on α; *scalar momentum* is the function $m|\alpha'|$ on I.

Newtonian particle A *Newtonian particle* is a curve $\alpha : I \longrightarrow E$ in Newtonian space, with I an interval in Newtonian time.

Newtonian scalar momentum *refer to* Newtonian momentum

Newtonian space *Newtonian space* is a Euclidean 3-space E, that is, a Riemannian manifold isometric to $\mathbf{R}^3$ (with dot product).

Newtonian spacetime The *Newtonian spacetime* is the Riemannian product manifold $\mathbf{R}^1 \times E$ of Newtonian time and Newtonian space.



nondegenerate form *refer to* definite form

nonspacelike geodesically complete *refer to* causal geodesically complete

nonspacelike geodesically incomplete *refer to* causal geodesically incomplete

norm in an Euclidean space Let $x \in \mathbf{R}^n$ with $x = (x_1, \ldots, x_n)$. Then the *norm of* x is defined by

$$\|x\| = \left[\sum_{i=1}^{n} (x_i)^2\right]^{\frac{1}{2}}.$$

norm in a semi-Riemannian manifold Let M be a semi-Riemannian manifold with metric $\mathbf{g}$ and $p \in M$. Then the *norm of* p *in* M is defined by

$$|p| = \mathbf{g}(p,p)^{\frac{1}{2}}.$$

norm on a vector space A *norm* on a vector space E is a mapping from E into the real numbers $\| \cdot \| : E \longrightarrow \mathbf{R}$ such that

i. $\| \cdot \| \geq 0$ for all $e \in E$ and $\|e\| = 0$ iff $e = 0$;

ii. $\|\lambda e\| = |\lambda| \|e\|$ for all $e \in E$ and $\lambda \in \mathbf{R}$;

iii. $\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|$ for all $e_1, e_2 \in E$.

normal bundle Let M^n be a semi-Riemannian submanifold of $\overline{M}^{n+k}$ and let NM be the set $\bigcup\{T_pM^{\perp} : p \in M\}$ of all normal vectors to M. Let $\pi : NM \longrightarrow M$ be the map carrying each $T_pM^{\perp}$ to $p \in M$. Then (NM, π) become a k-vector bundle over M and is called the *normal bundle of* M *in* $\overline{M}$.

normal connection The *normal connection* $D^{\perp}$ *of* $M \subset \overline{M}$ is the function $D^{\perp} : \mathfrak{X}(M) \times \mathfrak{X}(M)^{\perp} \longrightarrow \mathfrak{X}(M)^{\perp}$ given by $D^{\perp}_V Z = \mathrm{nor}\, \overline{D}_V Z$ for $V \in \mathfrak{X}(M)$, $Z \in \mathfrak{X}(M)^{\perp}$.
$D^{\perp}_V Z$ is called *normal covariant derivative* of Z with respect to V.

normal covariant derivative *refer to* normal connection

normal curvature tensor If $M \subset \overline{M}$ the function $\mathcal{R}^{\perp} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M)^{\perp} \longrightarrow \mathfrak{X}(M)^{\perp}$ given by

$$\mathcal{R}^{\perp}_{VW} X = D^{\perp}_{[V,W]} X - [D^{\perp}_{V}, D^{\perp}_{W}] X$$

is called the *normal curvature tensor of* $M \subset \overline{M}$.

normal curvature vector *refer to* umbilic

normal curvature vector field *refer to* totally umbilic

normal exponential map For a complete manifold M, the *normal exponential map*

$$\exp^{\perp} : NP \longrightarrow M$$

sends $v \in NP$ to $\gamma_v(1)$, where γ_v is the M geodesic of initial velocity v.
facta. $\exp^{\perp}$ carries radial lines in T_pP to geodesics of M normal to P at p.

normal parallel A vector field $Z \in \mathfrak{X}(M)^{\perp}$ is *normal parallel* provided $D^{\perp}_{V} Z = 0$ for all $V \in \mathfrak{X}(M)$.

normal space A topological space S is called *normal* if each two disjoint closed sets have disjoint neighborhoods.

nowhere dense subset *refer to* dense subset

n**-simplex** Let $\{a_0, \dots, a_k\}$ be a geometrically independent set in $\mathbf{R}^n$. Define the *n-simplex σ spanned by* $0, \dots, a_n$ to be the set

$$x = \sum_{i=0}^{k} t_i a_i \quad \text{where} \quad \sum_{i=0}^{k} t_i = 1$$

and $t_i \geq 0$ for all i. The numbers t_i are uniquely determined by x; they are called the *barycentric coordinates of the point x of σ with respect to* $a_0, \dots, a_k$. The points $a_0, \dots, a_k$ that span σ are called the *vertices of σ*

nullcone Let M be a Lorentz manifold. The *future [past] nullcone $\Lambda^+(p)$ of p in M* is the set of all points connected by a null curve; that is, $\Lambda^+(p) = \mathrm{bd}(J^+(p))$ $[\Lambda^-(p) = \mathrm{bd}(J^-(p))]$. The *nullcone $\Lambda(p)$ of p* is defined by

$$\Lambda(p) \equiv \Lambda^+(p) \cup \Lambda^-(p).$$

Sometimes it is called a *lightcone*.

null coordinate system A coordinate system u, v in a Lorentz surface is *null* provided its coordinate curves are null.
facta. The line element has the form $\mathrm{d}s^2 = 2F\mathrm{d}u$, where as usual $F = \langle \partial_u, \partial_v \rangle$.

null curve A curve α in M is *null* if all of its velocity vectors $\alpha'(s)$ are null. Sometimes α is called a *lightlike curve*.

null geodesically complete *refer to* lightlike geodesically complete

null geodesically incomplete *refer to* lightlike geodesically incomplete

null tangent vector A tangent vector $v \in V$ is *null* if $q(v) = 0$ but $v \neq 0$, for given symmetric bilinear form q. Sometimes v is called a *lightlike tangent vector*.

O

observer An *observer* in a Minkowski spacetime is just a material particle.

observer field An *observer field on an arbitrary spacetime* M is a timelike future pointing unit vector field U.
facta. 1. Each integral curve of U is indeed an observer, parametrized by proper time.
2. Observer fields are sometimes called *reference frames*.

one-form An *one-form* θ on a manifold M is a function that assigns to each point p an element θ_p of the cotangent space $T_p^\star M$
facta. θ assigns a number to every tangent vector .

one-parameter group of diffeomorphisms If X is a complete vector field with flow psi, then the set $\{\psi_t : t \in \mathbf{R}\}$ is a group of diffeomorphisms on M. This set is called a *one-parametergroup of diffeomorphisms*.

one-parameter subgroup A *one-parameter subgroup in a Lie group* G is a smooth homomorphism α from **R** (under addition) to G.

open cover *refer to* paracompact space

open neighborhood *refer to* topsp

open set *refer to* topsp

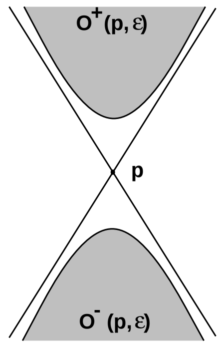
orbit *refer to* orbit manifold

orbit manifold Let Γ be a group of diffeomorphisms of a manifold M. For $p \in M$, the set $\{\phi(p) : \phi \in \Gamma\}$ is called the *orbit of* p *under* Γ. The collection of all such orbits is denoted by M/Γ. The natural map $k : M \longrightarrow M/\Gamma$ sends each point to its orbit under Γ. So M/Γ becomes a manifold and k a covering map. We call it a *orbit manifold* .

order of conjugacy Let $\mathcal{J}_{ab}$ be the set of all Jacobi fields on σ that vanish at a and b. Then $\mathcal{J}_{ab}$ is a subspace of the n-dimensional space consisting of those vanishing only at a. The dimension of $\mathcal{J}_{ab}$ is called the *order of conjugacy of* $\sigma(a)$ *and* $\sigma(b)$ *along* σ
cf. conjugate point

orientable manifold A manifold M is *orientable* provided that there exists a collection $\mathcal{O}$ of coordinate systems in M whose domains cover M and such that for each $\xi, \eta \in \mathcal{O}$, the Jacobian determinant function $J(\xi, \eta) = \det(\partial y^i/\partial x^j) > 0$. ($\mathcal{O}$ is called an *orientation atlas* for M.)

orientation atlas *refer to* orientable manifold



orthogonal vector Vecotrs $v, w \in V$ are *orthogonal*, written $v \perp w$, provided $\mathbf{g}(v, w) = 0$.

outer ball The *future outer ball* $O^+(p, \epsilon)$ [*past outer ball* $O^-(p, \epsilon)$] of $I^+(p)$ [$I^-(p)$] is given by

$$O^+(p, \epsilon) = \{q \in M : d(p, q) > \epsilon\}$$

and

$$O^-(p, \epsilon) = \{q \in M : d(q, p) > \epsilon\},$$

respectively.
facta. Since the Lorentzian distance function is lower semicontinuous where it is finite, the outer balls $O^+(p, \epsilon)$ and $O^-(p, \epsilon)$ are open in arbitrary spacetimes.

outer continuous Let $(M, \mathbf{g})$ be a spacetime . The set-valued function I^+ is said to be *outer continuous at* $p \in M$ if for each compact set $K \subset M - \overline{I^+(p)}$ there exists some neighborhood $U(p)$ of p such that $K \subset M - \overline{I^+(q)}$ for each $q \in U(p)$. Outer continuity of I^- may be defined dually.
cf. inner continuous

P

pair isometry Let M be a semi-Riemannian submanifold of $\overline{M}$. A *pair isometry from $M \subset \overline{M}$ to $N \subset \overline{N}$* is an isometry $\phi : \overline{M} \longrightarrow \overline{N}$ such that $\phi|_M$ is an isometry from M to N.
When $\overline{M} = \overline{N}$, ϕ is called a *congruent* from M to N.

paracompact space A collection $\{U_\alpha\}$ of subsets of M is a *cover of a set $W \subset M$* if $W \subset \bigcup U_\alpha$. It is an *open cover* if each U_α is open. A subcollection of them U_α which still covers is called a *subcover*. A *refinement $\{V_\beta\}$ of the cover $\{U_\alpha\}$* is a cover such that for each β, there is an α such that $V_\beta \subset U_\alpha$.
A topological space is *paracompact* if every open cover has an open locally finite refinement.

parallel tensor field A tensor field A is *parallel* provided its covariant differential zero, that is, $D_V A = 0$ for all $V \in \mathfrak{X}(M)$.

parallel translation For a curve $\alpha : I \longrightarrow M$ and $a, b \in I$, the function

$$P = P_a^b(\alpha) : T_pM \longrightarrow T_qM$$

sending each z to $Z(b)$ is called *parallel translation along α from $p = \alpha(a)$*.
facta. Parallel translation is a linear isometry.

parallel vector field A vector field V is *parallel* provided its covariant derivatives $D_X V$ are zero, for all $X \in \mathfrak{X}(M)$.
Let Z be a vector field on a curve $\alpha : I \longrightarrow M$. If $Z' = 0$, then Z is said to be *parallel*.

partially future imprisoned *refer to* future imprisoned

partition of unity A smooth *partition of unity* on a manifold M is a collection $\{f_\alpha : a \in A\}$ of functions $f_\alpha \in \mathcal{F}(M)$ such that

i. $0 \leq f_\alpha \leq 1$ for all $\alpha \in A$,
ii. $\{\mathsf{supp} f_\alpha : \alpha \in A\}$ is locally finite,
iii. $\sum_\alpha f_\alpha = 1$.

The partition of unity is said to be *subordinate* to an open covering $\mathcal{C}$ of M provided each set $\mathsf{supp} f_\alpha$ is contained in some element of $\mathcal{C}$. *facta.* The existence of partitions of unity subordinate to arbitrary open coverings is equivalent to the topological property *paracompactness*.

past Busemann function *refer to* Busemann function

past Cauchy development *refer to* Cauchy development

past Cauchy horizon *refer to* Cauchy horizon

past causal cut locus *refer to* future causal cut locus

past coray *refer to* future coray

past-directed *refer to* past pointing

past inner ball *refer to* inner ball

past nullcone *refer to* nullcone

past null cut locus *refer to* future null cut locus

past null cut point *refer to* future null cut point

past outer ball *refer to* outer ball

past pointing A tangent vector in a past causal cone is said to be *past pointing* (or *past-directed*). A causal curve is *past pointing* if all its velocity vectors are past pointing.

past pre-Busemann function *refer to* pre-Busemann function

past set *refer to* future set

past timelike cut locus *refer to* future timelike cut locus

past timelike cut point *refer to* future timelike cut locus

past-trapped *refer to* future-trapped

path A *path from p to q in a manifold M* is a continuous map $\alpha : I \longrightarrow M$ such that $\alpha(0) = p$ and $\alpha(1) = q$.

path product If $\alpha \in P(p, q)$ and $\beta \in P(q, r)$, let

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then $\alpha * \beta$ is a path from p to r is called the *path product of α to β*.

Penrose diagram A *Penrose diagram* is a two-diml representation of spherically symmetric spacetime . The radial null geodesics are represented by null geodesics at $\pm 45°$. Dotted lines represent the origin ($r = 0$) of polar coordinates . Points corresponding to smooth boundary points which are not singularities are represented by single lines. Double lines represent irremovable singularities.

perfect fluid A *perfect fluid on a spacetime M* is a triple (U, ρ, p) where :

i. U is a timelike future pointing unit vector field on M called the *flow vector field*

ii. $\rho \in \mathcal{F}(M)$ is the energy density function ; $p \in \mathcal{F}(M)$ is the *pressure function*

iii. The *stress-energy tensor* is

$$T = (\rho + p)U^\star \otimes U^\star + p\mathbf{g}$$

where $U^\star$ is the one-form metrically equivalent to U.
facta. For the stress-energy tensor T, the condition (iii) is equivalent to the following three equations for $X, Y \perp U$,

(a) $T(U,U) = \rho$,
(b) $T(X,U) = T(U,X) = 0$,
(c) $T(X,Y) = p\langle X,Y \rangle$.

period differential operator *refer to* linear differential operator

phase space of a system *refer to* mechanical system with symmetry

P**-Jacobi field** A Jacobi field V on a geodesic σ normal to P is the variation vector field of a variation $\mathbf{x}$ of σ through normal geodesics if and only if $V(0)$ is tangent to P and $\tan V(0) = \widetilde{II}(V(0), \sigma'(0))$. A Jacobi field satisfying these condition is called a *P-Jacobi field on σ*.

Poincaré-Cartan theorem Let X be a complete vector field a manifold M with flow $\psi(p,t)$ and let $\omega \in \Lambda^k(M)$. Then ω is an invariant k-form of X iff for all oriented compact k-manifolds with boundary $(V, \partial V)$ and smooth mappings $\phi : V \longrightarrow M$, we have

$$\int_V (\psi \circ \phi)^\star \omega = \int_V \phi^\star \omega$$

independent of t.

Poincaré duality Let X be a compact triangulated homology n-manifold. If X is orientable, then for all p, there is an isomorphism

$$H^p(X; G) \simeq H_{n-p}(X; G),$$

where G is an arbitrary coefficient group. If X is non-orientable, there is an isomorphism

$$H^p(X; \mathrm{Z}/2) \simeq H_{n-p}(X; \mathrm{Z}/2),$$

for all p.

Poincaré half-plane The *Poincaré half plane* P is the region $v > 0$ in $\mathbf{R}^3$, but with line element $\mathrm{d}s^2 = (\mathrm{d}u^2 + \mathrm{d}v^2)/v^2$.
facta. P has constant curvature $K = -1$.

Poincaré lemma Let U be the open unit ball in Euclidean space $\mathbf{R}^n$ and let $E^k(U)$ be the space of differential k-forms on U. Then for each $k \geq 1$, there is a linear transformation $h_k : E^k(U) \longrightarrow E^{k-1}(U)$ such that

$$h_{k+1} \circ \mathrm{d} + \mathrm{d} \circ h_k = \mathrm{id},$$

where d is the exterior differentiation.

Poisson bracket Suppose (M, ω) is a symplectic manifold and $\alpha, \beta \in \mathcal{X}^\star(M)$. The *Poisson bracket of* α *and* β is the one-form

$$\{\alpha, \beta\} = -\left[\alpha^\sharp, \beta^\sharp\right]^\flat.$$

For $f, g \in \mathcal{F}(M)$ with $X_f = (\mathrm{d}f)^\sharp \in \mathcal{X}(M)$. The *Poisson bracket of* f *and* g is the function

$$\{f, g\} = -i_{X_f} i_{X_g} \omega,$$

that is,

$$\{f, g\} = \omega(X_f, X_g).$$

facta. 1. $\{\alpha, \beta\} = -L_{\alpha^\sharp}\beta + L_{\beta^\sharp}\alpha + d(i_{alpha^\sharp} i_{\beta^\sharp}\omega)$.
2. $\{f, g\} = -L_{X_f} g = L_{X_g} f$.

polar map Let $L : T_o M \longrightarrow T_{\bar{o}}\overline{M}$ be a linear isometry, and let U be a normal neighborhood of o in M such that $\exp_{\bar{o}}$ is defined on the set $L(\exp_o^{-1}(U))$. Then the mapping

$$\phi_L = \exp_{\bar{o}} \circ L \circ \exp_o^{-1} : U \longrightarrow \overline{M}$$

is called the *polar map of* L *on* U.
facta. Polar maps always exist for U sufficiently small.

polyhedron *refer to* polytope

polytope Let K be a simplicial complex. Let $|K|$ be the subset of $\mathbf{R}^n$ that is the union of the simplices of K. Giving each simplex its natural topology as a subspace of $\mathbf{R}^n$, $|K|$ becomes a topological space and is called the *poloytope of* K or the *underlying space of* K. The space that is the polytope of a simplicial complex will be called a *polyheadron*.
facta. $|K|$ is Housdorff.

position vector field The *position vector field* $P \in \mathcal{X}(M)$ assigns to each point $p \in M$ the tangent vector $P_p \in T_p M$. (intuitively a duplicate, starting at p, of the arrow from 0 to p.)

positive definite form *refer to* definite form

positive semidefinite form *refer to* definite form

potential energy of a system *refer to* mechanical system with symmetry

pre-Busemann function Let $\gamma : I \longrightarrow M$ be a piecewise smooth timelike curve. For each $t \in I$, the *(future/past) pre-Busemann functions* $b^{\pm}_{\gamma,t} : J^{\mp}(\gamma(t)) \longrightarrow \mathbf{R} \cup \{\mp\infty\}$ of γ by

$$b^{+}_{\gamma,t}(p) = t - d(p, \gamma(t)),\ \textit{(future pre-Busemann function)}$$

$$b^{-}_{\gamma,t}(q) = t + d(\gamma(t), q).\ \textit{(past pre-Busemann function)}$$

rel. Busemann function

pregeodesic A *pregeodesic* is a smooth curve which may be reparametrized to be a geodesic.

presheaf A *presheaf* $P = \{S_U; \rho_{U,V}\}$ of K-modules on M consists of a K-module S_U for each open set U in M and a homomorphism $\rho_{U,V} : S_V \longrightarrow S_U$ for each inclusion $U \subset V$ of open sets in M such that $\rho_{U,V} =$ id and such that whenever $U \subset V \subset W$, the following diagram commutes:

$$\begin{tikzcd} S_W \arrow[r, "\rho_{V,W}"] \arrow[dr, "\rho_{U,W}"'] & S_V \arrow[d, "\rho_{U,V}"] \\ & S_U \end{tikzcd}$$

presheaf homomorphism Let $P = \{S_U; \rho_{U,V}\}$ and $P' = \{S'_U; \rho'_{U,V}\}$ be presheaves on M. A *presheaf homomorphism of P to P'* is a collection $\{\varphi_U\}$ of homomorphisms $\varphi : S_U \longrightarrow S'_U$ such that

$$\rho'_{U,V} \circ \varphi_V = \varphi_U \circ \rho_{U,V}$$

whenever $U \subset V$. A *presheaf isomorphism* is a presheaf homomorphism $\{\varphi_U\}$ in which each φ_U is an isomorphism of K-modules.

presheaf isomorphism *refer to* presheaf homomorphism

product neighborhood Let M be an n-manifold with boundary. Let's say bd$M$ has a *product neighborhood* in M if there is a homeomorphism

$$h : \text{bd}M \times [0, 1) \longrightarrow U$$

whose image of an open set in M, such that $h(x, 0) = x$ for each $x \in \text{bd}M$. *cf.* product topology

product topology Let S and T be topological spaces and $S \times T = \{(u, v) : u \in S$ and $v \in T\}$. The *product topology* on $S \times T$ consists of all subsets that are unions of sets of the form $U \times V$, where U is open in S and V is open in T. Thus these open rectangles form a basis for the topology.

projection The *projection* $\pi : TM \longrightarrow M$ is a map which sends $v \in T_pM$ to p.
facta. $\pi^{-1}(p) = T_pM$

projection on sheaf *refer to* sheaf

projective n-space Let's introduce an equivalence relation on the n-sphere S^n by defining $x \sim -x$ for each $x \in S^n$. The resulting quotient space is called *(real) projective n-space* and denoted P^n.

projective plane The *projective plane* P^2 is defined as the space obtained from the 2-sphere S^2 by identifying x with $-x$ for each $x \in S^2$.

proper face of simplex *refer to* abstract simplicial complex

properly discontinuous A group Γ of diffeomorphisms of a manifold M is *properly discontinuous* (and *acts freely*) provided

i. Each point $p \in M$ has a neighborhood U such that if $\phi(U)$ meets U for $\phi \in \Gamma$ then $\phi \in$ id.
ii. Points $p, q \in M$ not in the same orbit have neighborhoods U and V such that for every $\phi \in \Gamma$, $\phi(U)$ and V are disjoint.

facta. The deck transformation group of any covering is properly discontinuous.

proper time *refer to* material particle

proper time function *refer to* proper time synchronizable

proper time synchronizable field An observer field U o M is *proper time synchronizable* provided there exists a function $t \in \mathcal{F}(M)$ such that $U = -\mathrm{grad}\, t$. Then t is called a *proper time function* on M.

pseudometric *refer to* metric

pseudometric space *refer to* metric

pseudo n-manifold *refer to* relative pseudo n-manifold

pseudo-Riemannian manifold *refer to* semi-Riemannian manifold

pseudohyperbolic space Let $n \geq 2$ and $0 \leq \nu \leq n$. Then the *pseudohyperbolic space of radius $r > 0$ in $\mathbf{R}^{n+1}_\nu$* is the hyperquadric

$$\begin{aligned} H^n_\nu(r) &= q^{-1}(-r^2) \\ &= \left\{ p \in \mathbf{R}^{n+1}_\nu : \langle p, p \rangle = -r^2 \right\} \end{aligned}$$

with dimension n and index ν.

pseudosphere Let $n \geq 2$ and $0 \leq \nu \leq n$. Then the *pseudosphere of radius $r > 0$ in $\mathbf{R}^{n+1}_\nu$* is the hyperquadric

$$\begin{aligned} S^n_\nu(r) &= q^{-1}(r^2) \\ &= \left\{ p \in \mathbf{R}^{n+1}_\nu : \langle p, p \rangle = r^2 \right\} \end{aligned}$$

with dimension n and index ν.

pullback Let $\phi : M \longrightarrow N$ be a smooth mapping. If $A \in \mathcal{T}^0_s(N)$ with $s \geq 1$, let

$$(\phi^\star A)(v_1, \ldots, v_s) = A(\mathrm{d}\phi v_1, \ldots, \mathrm{d}\phi v_s)$$

for all $v_i \in T_p M$, $p \in M$. Then $\phi^\star(A)$ is called the *pullback of A by ϕ*.

Q

quadratic form The function $q : V \longrightarrow \mathbf{R}$ given by $q(v) = b(v, v)$ is the *associated quadratic form* of b.

quantizable manifold Let (P, ω) be a symplectic manifold. (P, ω) is *quantizable* if there is a principal circle bundle $\pi : Q \longrightarrow P$ over P and an one-form α on Q such that

i. α is invariant under the action of S';

ii. $\pi^{\star}\omega = \mathrm{d}\alpha$.

Often one calls α a connection and ω its curvature; Q is the *quantizing manifold*.

quantizing manifold *refer to* quantizable manifold

quasi-limit *Quasi-limits of future pointing curves in a manifold* is broken geodesics that are only approximate limits, their accuracy measured by a convex covering of M.
rel. limit sequence

quotient map A surjective map $p : X \longrightarrow Y$ is called a *quotient map* provided a subset U of Y is open if and only if the set $p^{-1}(U)$ is open in X.

quotient topology Let S be a topological space and $\sim$ an equivalence on S. Then

$$\{U \subset S/\sim : \pi^{-1}(U) \text{ is open in } S\}$$

is called the *quotient topology* on $S/\sim$.

R

rank of a free group *refer to* free abelian group

ray A *ray emanating from p* is the set of all points in a manifold of the form $p+tq$, where q is a fixed-point of $M-0$ and t ranges over the nonnegative reals.

real projective n-space *refer to* projective n-space

reduced cohomology group *refer to* cochain complex

reduced homology group *refer to* augmentation map

reductive manifold A coset manifold $M = G/H$ is *reductive* if there is an $\mathsf{Ad}(H)$-invariant subspace m of g that is complementary to h in g. Such m is called a *Lie subspace* for G/H.

reference frame *refer to* observer field

refinement of a cover *refer to* paracompact space

reflexive law *refer to* equivalence relation

regular boundary point Let $\mathrm{bd}(M)$ denote a boundary of a manifold M. A point $q \in \mathrm{bd}(M)$ is called a *regular boundary point of M* if there is a global extension $\widetilde{M}$ of M such that q may be naturally identified with a point of $\widetilde{M}$.
facta. A regular boundary point may be regarded as being a removable singularity of M.

regular curve A curve α is *regular* if $\alpha'(t) \neq 0$ for all t.

regular Lagrangian Let M be a manifold and $L \in \mathcal{F}(TM)$. If $\mathcal{F}L$ is regular at all points, then L is called a *regular Lagrangian*.

regular value A point $q \in N$ is called a *regular value* of a smooth mapping $\psi : M \longrightarrow N$ provided that $\mathrm{d}\psi_p$ is onto for every $p \in \psi^{-1}(q)$.

relative pseudo n-manifold A simplicial pair (K, K_0) is called a *rlative pseudo n-manifold* if :

i. The closure of $|K| - |K_0|$ equals a union of n-simplices.
ii. Each $(n-1)$-complex of K not in K_0 is a face of exactly two n-simplices of K.
iii. Given two n-simplices σ, σ' of K not in K_0, there is a sequence of n-simplices of K not in K_0

$$\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$$

such that $\sigma_i \cap \sigma_{i+1}$ is an $(n-1)$-simplex not in K_0, for each i.

If $K_0 = \emptyset$, K is simply called a *pseudo n-manifold*.

relative topology If A is a subset of a topological space S, the *relative topology on A* is defined by

$$\mathcal{O}_A = \{U \cap A : U \in \mathcal{O}\}.$$

restspace Let S be a spacelike hypersurface in M to which an observer field U is normal at every $p \in S$. Then the infinitesimal space $U_p^\perp$ is just T_pS for every $p \in S$, hence S is called *restspace of U*.

residual *refer to* Baire space

retract *refer to* retraction

retraction Let A be a subspace of a topological space X. A *retraction of X onto A* is a continuous map $r : X \longrightarrow A$ such that $r(a) = a$ for each $a \in A$. If there is a retraction map of X onto A, then A is called a *retract of X*.

reverse path Let α be a path from p to q, the *reverse path* $\overline{\alpha} \in P(q,p)$ is defined by $\overline{\alpha}(t) = \alpha(1-t)$.

reverse Schwarz inequality *refer to* backward Schwarz inequality

reverse triangle inequality *refer to* backward triangle inequality

Ricci curvature tensor Let $\mathcal{R}$ be the Riemannian curvature tensor of M. The *Ricci curvature tensor* Ric *of M* is the contraction $\mathsf{C}_3^1(\mathcal{R}) \in \mathcal{T}_2^0(M)$, whose components relative to a coordinate system are

$$\mathcal{R}_{ij} = \sum_m \mathcal{R}_{ijm}^m.$$

facta. By the symmetry of $\mathcal{R}$, the only nonzero contractions of $\mathcal{R}$ are $\pm\mathsf{Ric}$.

Ricci equation Let $\mathcal{R}^\perp : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M)^\perp \longrightarrow \mathfrak{X}(M)^\perp$ be the normal curvature function of $M \subset \overline{M}$. Then the *Ricci equation* is like that

$$\langle \mathcal{R}_{VW}^\perp X, Y \rangle = \langle \overline{\mathcal{R}}_{VW} X, Y \rangle + \langle \widetilde{II}(V,X), \widetilde{II}(W,Y) \rangle - \langle \widetilde{II}(V,Y), \widetilde{II}(W,X) \rangle,$$

where $X, Y \in \mathfrak{X}(M)^\perp$.

Ricci flat If the Ricci curvature tensor of a manifold M is identically zero, M is said to be *Ricci flat*.
facta. A flat manifold is Ricci flat.

Riemannian curvature tensor Let M be a semi-Riemannian manifold with the Levi-Civita connection D. The function $\mathcal{R} : \mathfrak{X}(M)^3 \longrightarrow \mathfrak{X}(M)$ given by

$$\mathcal{R}_{XY}Z = D_{[X,Y]}Z - [D_X, D_Y]Z$$

is a (1,3) tensor field on M called the *Riemannian curvature tensor of* M.

Riemannian distance For any points p and q of a connected Riemannian manifold M, the *Riemannian distance* $d(p,q)$ *from* p *to* q is the greatest lower bound of $\{L(\alpha) : \alpha \in \Omega(p,q)\}$, where $\Omega(p,q)$ is the set of all piecewise smooth curve segments in M from p to q.

Riemannian manifold A manifold with index $\nu = 0$ is called *Riemannian manifold*.

Riemannian metric For a connected Riemannian manifold M the Riemannian distance function $d : M \times M \longrightarrow \mathbf{R}$ is called the *Riemannian metric* on M provided for all $p, q, r \in M$:

i. $d(p,q) \geq 0$, and
$d(p,q) = 0$ if and only if $p = q$ (*positive definite*)

ii. $d(p,q) = d(q,p)$ (*symmetry*)

iii. $d(p,q) + d(p,r) \geq d(p,r)$ (*triangle inequality*).

facta. d is compatible with the topology of M.

Riesz representation theorem Let M be an orientable manifold with volume Ω. Let $\mathcal{B}$ denote the Borel sets of M, the σ-algebra generated by the open (or closed or compact) subsets of M. Then there is a unique measure μ_Ω on $\mathcal{B}$ such that for every continuous function of compact support,

$$\int f \mathrm{d}\mu_\Omega = \int_\Omega f.$$

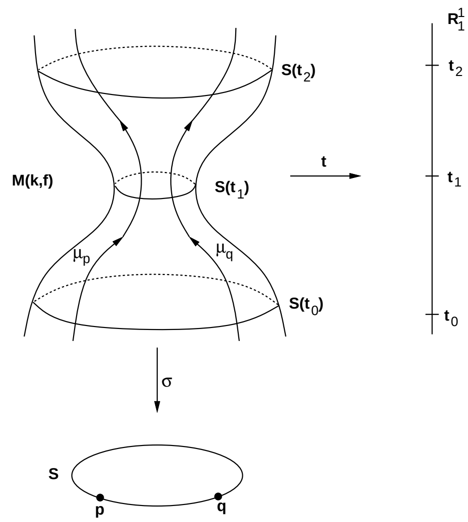
right-invariant *refer to* left-invariant

right-multiplication *refer to* left-invariant

Robertson-Walker spacetime Let S be a connected three-dimensional Riemannian manifold of constant curvature $k = -1, 0$, or 1. Let $f > 0$ be a smooth function on an open interval I in $\mathbf{R}^1_1$. Then the warped product

$$M(k,f) = I \times_f S$$

is called a *Robertson-Walker spacetime*.
facta. 1. $M(k,f)$ is the manifold $I \times S$ with the line element $-\mathrm{d}t^2 + f^2(t)\mathrm{d}\sigma^2$, where $\mathrm{d}\sigma^2$ is the line element of S.



2. Above f is called the *scale function* .
3. The Riemannian manifold S is called the *space* of $M(k, f)$.
4. Every plane containing a vector of flow vector field $U = \partial_t$ has curvature $K_u = f''/f$.
5. Every plane tangent to a spacelike slice has curvature

$$K_\sigma = ((f')^2 + k)/f^2.$$

6. We call the curvatures in (4) and (5) the *principal sectional curvatures* of $M(k, f)$.

S

saturated subset A subset C of X is *saturated with respect to a quotient map* p if it equals the complete inverse image $p^{-1}(A)$ of some subset A of Y.
rel. quotient map

saturation of a set Let $X^\star$ be a partition of X into closed sets and let $p : X \longrightarrow X^\star$ be the quotient map. For each closed set A of X, the set $p^{-1}(p(A))$ is called the *saturation of* A.

scalar curvature The *scalar curvature* S *of* M is the contraction $\mathsf{C}(\mathsf{Ric}) \in \mathcal{F}(M)$ of its Ricci tensor . In coordinates,

$$S = \sum \mathbf{g}^{ij}\mathcal{R}_{ij} = \sum \mathbf{g}^{ij}\mathcal{R}^k_{ijk}.$$

Contracting relative to a frame field yields

$$S = \sum_{i \neq j} K(E_i, E_j) = 2\sum_{i<j} K(E_i, E_j).$$

facta. $\mathrm{d}S = 2\mathsf{div}\mathsf{Ric}$.

scalar momentum in Minkowski spacetime *refer to* energy in Minkowski spacetime

scalar product A *scalar product* **g** *on a vector space* V is a nondegenerate symmetric bilinear form on V. Here V will denote a *scalar product space* .

scalar product space *refer to* scalar product

Schrödinger representation *refer to* full quantization

Schwarz inequality For vectors v, w in a vector space ,

$$|\langle v, w\rangle| \leq |v|\,|w|\,,$$

with equality if and only if v and w are independent, i.e., collinear.

Schwarzschild black hole *refer to* Schwarzschild exterior spacetime

Schwarzschild exterior spacetime For $\mathsf{M} > 0$, let P_I and P_{II} be the regions $r > 2\mathsf{M}$ and $0 < r < 2\mathsf{M}$ in the tr-half-plane $\mathbf{R}^1 \times \mathbf{R}^+$, each furnished with line element $-h\mathrm{d}t^2 + h^{-1}\mathrm{d}r^2$, where $h(r) = 1 - (2\mathsf{M}/r)$. If S^2 is the unit sphere, then the warped product $N = P_I \times_r S^2$ is called *Schwarzschild exterior spacetime* and $B = P_{II} \times S^2$ the *Schwarzschild black hole*, both of mass M.

Schwarzschild observer The integral curves α of observer field $U = \partial_t/\sqrt{h}$ are called *Schwarzschild observers*.

Schwarzschild spacetime The *Schwarzschild spacetime* is the spacetime with the following properties:

i. static;
ii. spherical symmetry;
iii. normalization;
iv. Minkowski at infinity and vacuum (i.e., Ricci flat).

Schwarzschild spherical coordinate system Let ϑ, φ be spherical coordinates on the unit sphere S^2. Let t, r be the usual Schwarzschild time and radius coordinates on $P_I \cup P_{II}$. The product coordinate system $(t, r, \vartheta, \varphi)$ in $U \cup B$ is called *Schwarzschild spherical coordinate system* .

second countable space *refer to* second countable topology

second countable topology Let S be a topological space. The topology is called *second countable* if it has a countable basis. Such a space S is called a *second countable space*.

second fundamental form The symmetric (0,2) tensor B metrically equivalent to the shape operator S is traditionally called the *second fundamental form of* $M \subset \overline{M}$.
(The *first fundamental form* is just the metric tensor of M.)
cf. shape tensor

section Let $\pi : E \longrightarrow B$ be a vector bundle. A C^r *section of* π is a map $\xi : B \longrightarrow E$ of class C^r such that for each $b \in B$, $\pi(\xi(b)) = b$. Let $\Gamma^r(\pi)$ denote the set of all C^r sections of π, together with the obvious real infinite-dimensional vector space structure.

sectional curvature Let Π be a nondegenerate tangent plane to M at p. For $v, w \in T_pM$, let $Q(v, w) = \langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2$. The number

$$K(v, w) = \langle \mathcal{R}_{vw} v, w\rangle / Q(v, w)$$

is independent of the choice of basis v, w for Π and is called the *sectional curvature* $K(\Pi)$ *of* Π.

section of sheaf *refer to* sheaf

semi-Euclidean space The space $\mathbf{R}^n_\nu$ is called *semi-Euclidean space* for the index $\nu \geq 0$. It gives a metric tensor

$$\langle v_p, w_p\rangle = -\sum_{i=1}^{\nu} v^i w^i + \sum_{j=\nu+1}^{n} v^j w^j,$$

for $v, w \in \mathbf{R}^n_\nu$. When $n \geq 2$, $\mathbf{R}^n_\nu$ is called the *Minkowski n-space* .
facta. When $\nu = 0$, $\mathbf{R}^n_\nu$ is the Euclidean n-space.

semiorthogonal group Let $GL(n, \mathbf{R})$ be a n-dimensional general linear group on $\mathbf{R}$ and ε the signature matrix. A *semiorthogonal group* $O(\nu, n-\nu)$ is the group of all matrices $g \in GL(n, \mathbf{R})$ that preserve the scalar product $\langle v, w \rangle = \varepsilon v \cdots w$ of $\mathbf{R}^n_\nu$.
facta. 1. It is the same as the set of all linear isometries $\mathbf{R}^n_\nu \longrightarrow \mathbf{R}^n_\nu$.
2. $O(\nu, n-\nu)$ is closed subgroup of $GL(n, \mathbf{R})$ and hence is itself a Lie group.

semi-Riemannian covering map A *semi-Riemannian covering map* $k : \widetilde{M} \longrightarrow M$ is a covering map of semi-Riemannian manifolds that is a local isometry.

semi-Riemannian group Sometimes a Lie group furnished with a bi-invariant metric is called a *semi-Riemannian group*.

semi-Riemannian manifold A *semi-Riemannian manifold* is a smooth manifold M furnished with a metric tensor $\mathbf{g}$. It is often called *pseudo-Riemannian manifold*.

semi-Riemannian submanifold Let P be a submanifold of a semi-Riemannian manifold M. If the pullback $j^\star(\mathbf{g})$ is a metric tensor on P it makes P a *semi-Riemannian submanifold of M*.

semi-Riemannian submersion A *semi-Riemannian submersion* $\pi : M \longrightarrow B$ is a submersion of semi-Riemannian manifolds such that

i. The fibers $\pi^{-1}(b), b \in B$, are semi-Riemannian submanifolds of M,
ii. dπ preserves scalar product of vectors normal to fibers.

facta. Since the fibers of a submersion are smooth submanifolds, the condition (i) is automatically true if M is Riemannian.

separation For points p, q in a Minkowski spacetime M, the number $pq = |\vec{pq}| \geq 0$ is called the *separation between p and q*.

serpent lemma Given a homomorphism of short exact sequences of abelian groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0, \end{array}$$

there is an exact sequence

$$\begin{aligned} 0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \\ \longrightarrow \mathrm{cok}\alpha \longrightarrow \mathrm{cok}\beta \longrightarrow \mathrm{cok}\gamma \longrightarrow 0. \end{aligned}$$

This property is called the *serpent lemma*.

set of labels *refer to* labelling of vertices

shape operator Let U be a unit normal vector on a semi-Riemannian hypersurface $M \subset \overline{M}$. The (1,1) tensor field S on M such that

$$\langle S(V), W\rangle = \langle II(V, W), U\rangle$$

for all $V, W \in \mathfrak{X}(M)$, is called the *shape operator of* $M \subset \overline{M}$ *derived from* U.
facta. A semi-Riemannian hypersurface $M \subset \overline{M}$ is totally umbilic if and only if its shape operator is scalar.
rel. shape tensor

shape tensor The function $II : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)^{\perp}$ such that

$$II(V, W) = \text{nor } \overline{D}_V W$$

is $\mathcal{F}(M)$-bilinear and symmetric. II is called the *shape tensor* (or *second fundamental form*) of $M \subset \overline{M}$.
facta. On $\mathbf{R}^n_\nu$, isometric submanifolds are congruent if and only if they have the same shape tensor .
cf. second fundamental form

sharp operator *refer to* flat operator

sheaf A *sheaf* $\mathcal{S}$ *of* K*-module over* M consists of a topological space $\mathcal{S}$ together with a map $\pi : \mathcal{S} \longrightarrow M$ satisfying

i. π is a local homeomorphism of $\mathcal{S}$ onto M.
ii. $\pi^{-1}(m)$ is a K-module for each $m \in M$.
iii. The composition laws are continuous in the topology on $\mathcal{S}$.

The map π is called the *projection* and the K-module $S_m = \pi^{-1}(m)$ is called the *stalk* over $m \in M$. A continuous map $f : U \longrightarrow \mathcal{S}$ such that $\pi \circ f = \text{id}$ is called *section* of $\mathcal{S}$ over an open set $U \subset M$.

sheaf cohomology theory A *sheaf cohomology theory* $\mathcal{H}$ *for a manifold* M *with coefficients in sheaves of* K*-modules over* M consists of

i. a K-modules $H^q(M, \mathcal{S})$ for each sheaf $\mathcal{S}$ and for each integer q;
ii. a homomorphisms $H^q(M, \mathcal{S}) \longrightarrow H^q(M, \mathcal{S}')$ for each homomorphism $\mathcal{S} \longrightarrow \mathcal{S}'$ and for each integer q;
iii. a homomorphism $H^q(M, \mathcal{S}'') \longrightarrow H^{q+1}(M, \mathcal{S}')$ for each short exact sequence $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ and for each integer q,

such that the following properties hold:

(a) $H^q(M,\mathcal{S})=0$ for $q\geq 0$ and there is an isomorphism $H^0(M,\mathcal{S})\simeq\Gamma(\mathcal{S})$ such that for each homomorphism $\mathcal{S}\longrightarrow\mathcal{S}'$, the diagram

$$\begin{array}{ccc} H^0(M,\mathcal{S}) & \simeq & \Gamma(\mathcal{S}) \\ \downarrow & & \downarrow \\ H^0(M,\mathcal{S}') & \simeq & \Gamma(\mathcal{S}') \end{array}$$

commutes.

(b) $H^q(M,\mathcal{S})=0$ for all $q>0$ if $\mathcal{S}$ is a fine sheaf.

(c) If $0\to\mathcal{S}'\to\mathcal{S}\to\mathcal{S}''\to 0$ is exact, then the following is exact:

$$\cdots\to H^q(M,\mathcal{S})\to H^q(M,\mathcal{S})\to H^q(M,\mathcal{S}'')\to H^{q+1}(M,\mathcal{S}')\to\cdots$$

(d) The identity homomorphism id : $\mathcal{S}\longrightarrow\mathcal{S}$ induces the identity homomorphism id : $H^q(M,\mathcal{S})\longrightarrow H^q(M,\mathcal{S})$.

(e) If the diagram

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathcal{S}' \\ & \searrow & \downarrow \\ & & \mathcal{S}'' \end{array}$$

commutes, then for each q, so does the diagram

$$\begin{array}{ccc} H^q(M,\mathcal{S}) & \longrightarrow & H^q(M,\mathcal{S}') \\ & \searrow & \downarrow \\ & & H^q(M,\mathcal{S}'') \end{array}$$

(f) For each homomorphism of short exact sequences of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}' & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{S}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}' & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{T}'' & \longrightarrow & 0 \end{array}$$

the following diagram commutes:

$$\begin{array}{ccc} H^q(M,\mathcal{S}'') & \longrightarrow & H^{q+1}(M,\mathcal{S}') \\ \downarrow & & \downarrow \\ H^q(M,\mathcal{T}'') & \longrightarrow & H^{q+1}(M,\mathcal{T}') \end{array}$$

The module $H^q(M,\mathcal{S})$ is called the *q-th cohomology module of M with coefficients in the sheaf $\mathcal{S}$ relative to the cohomology theory $\mathcal{H}$.*

sheaf homomorphism *refer to* sheaf mapping

sheaf isomorphism *refer to* sheaf mapping

sheaf mapping Let $\mathcal{S}$ and $\mathcal{S}'$ be sheaves on M with projections π and π', respectively. A continuous map $\varphi:\mathcal{S}\longrightarrow\mathcal{S}'$ such that $\pi'\circ\varphi=\pi$ is called a *sheaf mapping*. Obviously, sheaf mappings are necessarily local homeomorphisms and they map stalks to stalks. A sheaf mapping φ which is a homomorphism of K-modules on each stalk is called a *sheaf homomorphism*. A *sheaf iso* is a sheaf homomorphism with an inverse which is also a sheaf homomorphism.

short exact sequence A sequence of three groups and homomorphisms is called a *short exact sequence* if

$$0 \longrightarrow A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \longrightarrow 0$$

is exact.

signature Let g be a scalar product on a vector space V. Let $e_1, \ldots, e_n$ be an orthogonal basis for V.

$$\mathbf{g}(e_i, e_j) = \delta_{ij}\varepsilon_j,$$

where $\varepsilon_j = \mathbf{g}(e_j, e_j) = \pm 1$ and δ_{ij} is the Kronecker delta. The *signature for V* is $(\varepsilon_1, \ldots, \varepsilon_n)$.

signature matrix For $0 \leq \nu \leq n$, the *signature matrix ε* is the diagonal matrix $(\delta_{ij}\varepsilon_j)$ whose diagonal entries are $\varepsilon_1 = \cdots + \varepsilon_\nu = -1$ and $\varepsilon_{\nu+1} = \cdots = \varepsilon_n = +1$.
facta. $\varepsilon^{-1} = \varepsilon =^t \varepsilon$, where ${}^t g$ denotes the transpose of g.

sign of semi-Riemannian hypersurface The *sign ε of semi-Riemannian hypersurface M of $\overline{M}$* is

$$\begin{cases} +1 & \text{if } \langle z, z\rangle > 0, \\ -1 & \text{if } \langle z, z\rangle < 0, \end{cases}$$

for every normal vector $z \neq 0$.
facta. For a Riemannian manifold , every hypersurface of Riemannian with sign $+1$.

sign of the permutation *refer to* alternating multilinear map

simplex *refer to* abstract simplicial complex
cf. n-simplex

simplicial approximation Let $h : |K| \longrightarrow |L|$ be a continuous map. If $f : K \longrightarrow L$ is a simplicial map such that

$$h(\text{St } v) \subset \text{St } f(v),$$

for each vertex v of K, then f is called a *simplicial approximation to h.*
rel. star of a vertex

simplicial complex A *simplicial complex K in* $\mathbf{R}^n$ is a collection of simplices in $\mathbf{R}^n$ such that

i. Every face of a simplex of K is in K.

ii. The intersection of any two simplices of K is a face of each of them.

simplicial homeomorphism *refer to* simplicial map

simplicial map Let K and L be complexes and let $f : K^{(0)} \longrightarrow L^{(0)}$ be a map. Whenever the vertices $v_0, \ldots, v_n$ of K span a simplex of K, the points $f(v_0), \ldots, f(v_n)$ are vertices of a simplex of L. Hence f can be extended to a continuous map $g : |K| \longrightarrow |L|$ such that

$$x = \sum_{i=0}^{n} t_i v_i \text{ implies } g(x) = \sum_{i=1}^{n} t_i f(v_i).$$

The map g is called the *(linear) simplicial map induced by the vertex map* f. If the vertex map f is an one-to-one correspondence, g becomes a homeomorphism; that is called a *simplicial homeomorphism of* K *with* L.

simply connected manifold A manifold M is *simply connected* provided M is connected and its fundamental group is trivial, that is, reduces to the identity element .

singular point *refer to* critical point

skeleton Let K be a simplicial complex. A subcomplex of K such that the collection of all simplices of K of dimension at most p is called the p-*skeleton of* K and is denoted by $K^{(p)}$. The points of collection $K^{(0)}$ are called the *vertices of* K.

skew-symmetric tensor field *refer to* symmetric tensor field

slice Let (U, ϕ) be a coordinate system on M with coordinate functions $x_1, \ldots, x_d$ and c an integer with $0 \leq c \leq d$. Let $a \in \varphi(U)$ and let

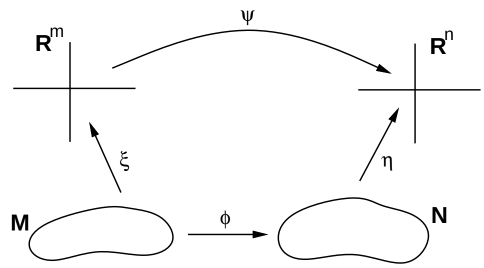
$$S = \{q \in U : x_i(q) = r_i(a), i = c+1, \ldots d\}.$$

The subspace S of M together with the coordinate system $\{x_j\}_{1 \leq j \leq c}$ on S forms a submanifold of M which called a *slice* of the coordinate system (U, φ).

smooth distribution *refer to* distribution

smooth function on Euclidean space A function ϕ from an open set U in $\mathbf{R}^n$ is *smooth* if each real-valued function $u^i \circ \phi$ is smooth, where u^i is the natural coordinate function ($1 \leq i \leq n$).

smooth function on manifold A function $\phi : M^m \longrightarrow N^n$ is *smooth* provided that for every coordinate system ξ in M and η in N, the coordinate expression $\psi \equiv \eta \circ \phi \circ \xi^{-1}$ is Euclidean smooth.



smooth manifold A *smooth manifold* M is a Housdorff space furnished with a complete atlas. Simply, it is called a *manifold*.
facta. 1. A manifold is second countable locally Euclidean space.
2. Euclidean spaces $\mathbf{R}^n$ are manifolds.
3. The spheres $S^n = \{a \in \mathbf{R}^{n+1} : |a| = 1\}$ are manifolds.

smooth one-form Let θ be a *smooth one-form* on M if θX is smooth for all $X \in \mathfrak{X}(M)$.

smooth vector field A vector field V is *smooth* if Vf is smooth for all $f \in \mathcal{F}(M)$.

space form A *space form* is a complete connected semi-Riemannian manifold of constant curvature .
facta. 1. Simply connected space forms are isometric if and only if they have the same dimension, index and constant curvature.

spacelike curve A curve α in M is *spacelike* if all of its velocity vectors $\alpha'(s)$ are spacelike.

spacelike geodesically complete A semi-Riemannian manifold M is said to be *spacelike geodesically complete* if all spacelike inextendible geodesics are complete.

spacelike geodesically incomplete A semi-Riemannian manifold M is said to be *spacelike geodesically incomplete* if some spacelike geodesic is incomplete.

spacelike tangent vector A tangent vector v to M is *spacelike* if $q(v,v) > 0$ or $v = 0$, where q is a symmetric bilinear form.

spacetime A *spacetime* $(M, \mathbf{g})$ is a connected time-oriented four-dimensional Lorentz manifold. (Informally, *time-oriented* is often weakened to *time-orientable*.)

spatial pressure gradient *refer to* force equation

special linear group Let K be a field. The *special linear group* $SL(n,K)$ is the multiplicative group of all $n \times n$ *unimodular* matrices over K; that is,

det $a = -1$ for all $a \in SL(n, K)$.
facta. For the determinant function det $: GL(n, \mathbf{R}) \longrightarrow \mathbf{R} - \{0\}$, the kernel of det is $SL(n, \mathbf{R})$, thus $SL(n, \mathbf{R})$ is a closed subgroup of $GL(n, \mathbf{R})$.
rel. general linear group

split sequence Consider a short exact sequence

$$0 \longrightarrow A_1 \xrightarrow{\phi} A_2 \xrightarrow{\psi} A_3 \longrightarrow 0.$$

This sequence is called *split* if the group $\phi(A_1)$ is a direct summand in A_2; that is, $A_2 = \phi(A_1) \oplus B$ for some subgroup B of A_2.

stable point Let p be a critical point of vector field X. Then p is *stable* (or, *Lyapunov stable*) if for any neighborhood U of p, there is a neighborhood V of p such that if $q \in V$, then p is complete to positive direction abd the flow $\psi_\lambda(m) \in U$ for all $\lambda \geq 0$.

stably causal Let $Lor(M)$ denote the space of all Lorentz metrics on M. A spacetime $(M, \mathbf{g})$ is said to be *stably causal* if there is a fine C^0 neighborhood $U(\mathbf{g})$ of $\mathbf{g}$ in $Lor(M)$ such that each $\mathbf{h} \in U(\mathbf{g})$ is causal.

stalk *refer to* sheaf

standard fiber *refer to* vector bundle

standard metric *refer to* Euclidean metric

standard static spacetime Let S be a 3-dimensional Riemannian manifold , I an open interval, and $g > 0$ a smooth function on S. Let t and σ as usual be the projections of $I \times S$ onto I and S. The *standard static spacetime* $I_g \times S$ is the manifold $I \times S$ with line element

$$-g(\sigma)^2 \mathrm{d}t^2 + \mathrm{d}s^2,$$

where $\mathrm{d}s^2$ is the lift of the line element of S.

star condition Let $h : |K| \longrightarrow |L|$ be a continuous map. Let's say that h satisfies the *star condition with respect to K and L* if for each vertex v of K, there is a vertex w of L such that

$$h(\mathrm{St}\, v) \subset \mathrm{St}\, w.$$

star in Schwarzschild spacetime The *star* is assumed to be static and spherically symmetric and to be the only source of gravitation for the spacetime.

star of a simplex Let s be a simplex of the complex K. The *star of s in K*, denoted by St s, is the union of the interiors of all simplices of K having s as a face. Its closure, denoted by $\overline{\mathrm{St}}\, s$, is called the *closed star of s in K*. The *link of s in K*, denoted Lk s, is the union of all simplices of K lying in $\overline{\mathrm{St}}\, s$ that are disjoint from s.

star of a vertex If v is a vertex of a simplicial complex K, the *star of v in K*, denoted by St v or St (v, K), is the union of the interiors of those simplices of K that have v as a vertex. Its closure, denoted by $\overline{\text{St}}\,v$, is called the *closed star of v in K*. The set $\overline{\text{St}}\,v - \text{St}\,v$ is called the *link of v in K* and is denoted by Lk v.

starshaped space A subset $\mathcal{S}$ of a vector space is *starshaped about 0* if $v \in \mathcal{S}$ implies $tv \in \mathcal{S}$ for all $0 \leq t \leq 1$.

static spacetime A spacetime M is *static* relative to an observer field U provided U is irrotational and there is a smooth function $g > 0$ on M such that gU is a Killing vector field .

Steenrod five-lemma Suppose one is given a commutative diagram of abelian groups and homomorphisms :

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

where the horizontal sequences are exact. If f_1, f_2, f_4 and f_5 are isomorphisms, so is f_3.

Stokes' theorem Let M be an oriented smooth n-manifold with boundary and $\alpha \in \Lambda^{n-1}(M)$ have compact support. Let $i : \partial M \longrightarrow M$ be the inclusion map such that $i^\star\alpha \in \Lambda^{n-1}(\partial M)$. Then

$$\int_{\partial M} i^\star \alpha = \int_M \mathrm{d}\alpha$$

or, in brief form,

$$\int_{\partial M} \alpha = \int_M \mathrm{d}\alpha.$$

Above $\star$ is the Hodge star operator.

stress-energy tensor *refer to* perfect fluid

strong causality condition The *strong causality condition* holds at $p \in M$ provided that given any neighborhood U of p, there is a neighborhood $V \subset U$ of p such that every causal curve segment with endpoints in V lies entirely in U.

strong energy condition In terms of the stress-energy tensor of a manifold M, the timelike convergence condition becomes

$$T(u, u) \geq \frac{1}{2}\mathsf{C}(T)\langle u, u\rangle$$

for all timelike (and null) tangent vectors to M. This condition is called the *strong energy condition on M*.

subcomplex *refer to* abstract simplicial complex

subcover *refer to* paracompact space

submanifold A manifold $\mathcal{S}$ is a *submanifold* of a manifold M provided:

i. $\mathcal{S}$ is a topology subspace of M
ii. The inclusion map $\jmath : \mathcal{S} \subset M$ is smooth and its differential map $\mathrm{d}\jmath$ is one-to-one at each point $p \in \mathcal{S}$.

submersion A mapping $\psi : M \longrightarrow B$ is a *submersion* provided that ψ is a smooth mapping onto B such that $\mathrm{d}\psi_p$ is onto for all $p \in M$.

subordinate to a covering *refer to* partition of unity

subsheaf An open set $\mathcal{B}$ in the sheaf $\mathcal{S}$ such that the subset $\mathcal{B}_m = \mathcal{B} \cap \mathcal{S}_m$ is a submodule of $\mathcal{S}_m$ for each $m \in M$ is called a *subsheaf of* $\mathcal{S}$.

support The *support* of $f \in \mathcal{F}(M)$ is the closure of the set $\{p \in M : f(p) \neq 0\}$ and denoted by $\mathsf{supp} f$.

suspension of a complex Let K be a complex and let $w_0 * K$ and $w_1 * K$ be two cones on K whose polytopes intersect in $|K|$ alone. Then

$$S(K) = (w_0 * K) \cup (w_1 * K)$$

is a complex and is called a *suspension of* K.
facta. Given K, the complex $S(K)$ is uniquely defined up to a simplicial isomorphism.

suspension of a space Let X be a space. The *suspension of* X is the quotient space of $X \times 1$ to a point, and the subset $X \times (-1)$ to a point. It is denoted $S(X)$. $w_0 * K$ and $w_1 * K$ be two cones on K whose polytopes intersect in $|K|$ alone. Then

$$S(K) = (w_0 * K) \cup (w_1 * K)$$

is a complex and is called a *suspension of* K.
facta. Given K, the complex $S(K)$ is uniquely defined up to a simplicial isomorphism.

symmetric law *refer to* equivalence relation

symmetric space A *semi-Riemannian symmetric space* is a connected semi-Riemannian manifold M such that for each $p \in M$, there is a (unique) symmetry $\zeta_p : M \longrightarrow M$ with differential map $-\mathrm{id}$ on T_pM. The isometry ζ_p is called the *global isometry of M at p.*
facta. 1. Symmetry implies local symmetry.
2. $\mathbf{R}^n$, S^n are symmetric .
3. Every connected hyperquadric is symmetric .

symmetric tensor field Let A be a covariant or contravariant tensor of type at least 2. A is *symmetric* if transposing any two of its argument leaves its value unchanged. A is *skew-symmetric* if each such reversal produces a sign change.

symplectic eigenvalue theorem Suppose (E, ω) is a symplectic vector space, $f \in \mathrm{Sp}(E, \omega)$ and λ is an eigenvalue of f of multiplicity k. Then $1/\lambda$ is an eigenvalue of f of multiplicity k. Moreover, the multiplicities of the eigenvalues $+1$ and -1 are even if they occur.

symplectic form on a manifold *refer to* symplectic manifold

symplectic form on a vector space *refer to* symplectic vector space

symplectic group Let (E, ω) be a symplectic vector space. Then the set of all symplectic mappings $f : E \longrightarrow E$ forms a group under composition. It is called the *symplectic group*, denoted by $\mathrm{Sp}(E, \omega)$.

symplectic manifold A *symplectic form* (or a *symplectic structure*) on a manifold M is a nondegenerate closed two-form ω on M. A *symplectic manifold* (M, ω) is a manifold M together with a symplectic form ω on M.

symplectic map *refer to* symplectic vector space

symplectic structure *refer to* symplectic manifold

symplectic vector space A *symplectic form* on a vector space E is a nondegenerate two-form $\omega \in \Lambda^2(E)$. The pair (E, ω) is called a *symplectic vector space*. If (E, ω) and (E, ρ) are symplectic vector spaces, a linear map $f : E \longrightarrow F$ is *symplectic* if $f^\star \rho = \omega$.

synchronizable field An observer field U on M is *synchronizable* provided there are smooth function $h > 0$ and t on M such that $U = -h \operatorname{grad} t$.
facta. A synchronizable observer field is irrotational, since U is normal to the level hypersurfaces of t, which are restspaces.
rel. restspace

Synge's formula for second variation Let $\sigma : [a, b] \longrightarrow M$ be a geodesic sequence of speed $c > 0$ and sign ε. If **x** is a variation of σ, then

$$L''(0) = \frac{\varepsilon}{c} \int_a^b \left\{ \langle \overset{\perp}{V}{}', \overset{\perp}{V}{}' \rangle - \langle R_{V\sigma'} V, \sigma' \rangle \right\} \mathrm{d}u + \frac{\varepsilon}{c} \langle \sigma', A \rangle \Big|_a^b,$$

where V is the variation vector field , A the transverse acceleration vector field of **x**.

T

tangent bundle For a manifold M, let *tangent bundle TM of M* be the set $\bigcup\{T_pM : p \in M\}$ of all tangent spaces to M.

tangent space The *tangent space to M at p* is the set of all tangent vectors to M at p and denoted by T_pM.

tangent vector The *tangent vector to M at p* is the real-valued function $v : \mathcal{F}(M) \longrightarrow \mathbf{R}$ that is

i. $\mathbf{R}$-linear : $v(af + bg) = a(v(f) + bv(g)$,
ii. Leibnizian : $v(fg) = v(f)g + fv(g)$, for $a, b \in \mathbf{R}$ and $f, g \in \mathcal{F}(M)$.

Taylor's theorem A map $f : U \subset E \longrightarrow F$ is a class C^r iff there are continuous mappings

$$\begin{aligned} \varphi_p : U \subset E &\longrightarrow L^p_s(E, F),\ p = 1, \dots, r \\ R : \widetilde{U} &\longrightarrow L^r_s(E, F), \end{aligned}$$

where $\widetilde{U}$ is a thickening of U such that for all $(u, h) \in \widetilde{U}$,

$$f(u + h) = f(u) + \frac{\varphi_1(u)}{1!}h + \frac{\varphi_2(u)}{2!}h^2 + \cdots + \frac{\varphi_r(u)}{r!}h^r + R(u, h)h^r,$$

where $h^r = (h, \dots, h)$ (r-times) and $R(u, 0) = 0$.

tensor For integers $r, s \geq 0$ not both zero, a K-multilinear function

$$A : (V^\star)^r \times V^s \longrightarrow K$$

is called a *tensor of type (r,s) over V*. The set of all tensors of type (r,s) over V is denoted by $\mathcal{T}^r_s(V)$ and $\mathcal{T}^r_s(V)$ is a module over K.
facta. $\mathcal{T}^0_0(M) = \mathcal{F}(M)$.

tensor derivation A *tensor derivation $\mathcal{D}$ on a smooth manifold M* is a set of $\mathbf{R}$-linear functions

$$\mathcal{D} = \mathcal{D}^r_s : \mathcal{T}^r_s(M) \longrightarrow \mathcal{T}^r_s(M)\ \ (r, s \geq 0)$$

such that for any tensors A and B,

i. $\mathcal{D}(A \otimes B) = \mathcal{D}A \otimes B + A \otimes \mathcal{D}B$,
ii. $\mathcal{D}(\mathsf{C}A) = \mathsf{C}(\mathcal{D}A)$, for every contraction C.

facta. $\mathcal{D}$ is $\mathbf{R}$-linear.

tensor field A *tensor field A on a manifold M* is a tensor over the $\mathcal{F}(M)$-module $\mathfrak{X}(M)$, such that

$$A : \mathfrak{X}^\star(M)^r \times \mathfrak{X}(M)^s \longrightarrow \mathcal{F}(M)$$

with $(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s) \longmapsto f$. Here θ^i occupies the *i*-th contravariant slot, X_j the *j*-th covariant slot of A.

terminal indecomposable future set A *terminal indecomposable future set TIF* is a subset S of M such that

i. S is an indecomposable future set.
ii. S is not the chronological future of any point $p \in M$.

terminal indecomposable past set A *terminal indecomposable past set TIP* is a subset S of M such that

i. S is an indecomposable past set.
ii. S is not the chronological past of any point $p \in M$.

test particle A particle whose energy-momentum makes negligible contribution to the stress-energy tensor is called a *test particle*.

thickening of a set Let $U \subset E$ be an open set. As $+ : E \times E$ with

i. $U \times \{0\} \subset \widetilde{U}$;
ii. $u + \xi h \in U$ for all $(u, h) \in \widetilde{U}$ and $0 \leq \xi \leq 1$;
iii. $(u, h) \in \widetilde{U}$ implies $u \in U$.

The set

$$\widetilde{U} = \left\{(+)^{-1}(U)\right\} \bigcap \{U \times E\}$$

is called a *thickening of U*.

tidal force operator For a vector $0 \neq v \in T_pM$, the *tidal force operator F_v* : $v^\perp \longrightarrow v^\perp$ is given by $F_v(y) = \mathcal{R}_{yv}v$.
facta. F_v is self-adjoint linear operator on $v^\perp$ and $\mathsf{trace}F_v = -\mathsf{Ric}(v, v)$.

time average *refer to* Mean Ergodic theorem

timecone Let T be the set of timelike vectors in a Lorentz vector space V. For $u \in T$,

$$C(u) = \{v \in T : \langle u, v \rangle < 0\}$$

is the *timecone of V containing u*. The *opposite* timecone is

$$C(-u) = -C(u) = \{v \in T : \langle u, v \rangle > 0\}.$$

facta. 1. Since $u^\perp$ is spacelike , T is the disjoint union of these two timecones.
2. Timecones are convex.

timelike Cauchy complete The causal spacetime $(M, \mathbf{g})$ is called *timelike Cauchy complete* if any sequence $\{p_n\}$ of points with $p_n \ll p_{n+m}$ for $1 \leq n, m < \infty$ and $d(p_n, p_{n+m}) \leq B_n$ [or else $p_{n+m} \ll p_n$ and $d(p_{n+m}, p_n) \leq B_n$] for all $m \geq 0$, where $B_n \to 0$ as $n \to \infty$, is a convergent sequence .

timelike curve A curve α in M is *timelike* if all of its velocity vectors $\alpha'(s)$ are timelike.

timelike convergence condition For all timelike tangent vectors to M, the *timelike convergence condition* is

$$\mathsf{Ric}(u, u) \geq 0.$$

It says that, on average, gravity attracts.

timelike geodesically complete A semi-Riemannian manifold M is said to be *timelike geodesically complete* if all timelike inextendible geodesics are complete.

timelike geodesically incomplete A semi-Riemannian manifold M is said to be *timelike geodesically incomplete* if some timelike geodesic is incomplete.

timelike tangent vector A tangent vector v to M is *timelike* if $q(v, v) < 0$, where q is a symmetric bilinear form.

time-orientable If a manifold M admits a time-orientation, M is said to be *time-orientable*.

time-orientation Let τ be a smooth function on M that assigns to each point p a timecone τ_p in T_pM, that is, for each $p \in M$ there is a smooth vector field V on some neighborhood U of p such that $V_q \in \tau_q$ for each $q \in U$. The τ is called a *time-orientation of* M.

time separation If $p, q \in M$, the *time separation* $\tau(p, q)$ from p to q is

$$\sup \{L(\alpha) : \alpha \text{ is future pointing causal curve segment from } p \text{ to } q\}.$$

The *time separation* $\tau(A, B)$ of subsets A and B of M is

$$\sup \{\tau(a, b) : a \in A, b \in B\}.$$

facta. τ need not be continuous.

topological group A *topological group* G is an abstract group G which has a topology such that the map $(\sigma, \tau) \mapsto \sigma\tau^{-1}$ of $G \times G \longrightarrow G$ is continuous.

topological hypersurface A subsp S of a topological manifold T is a *topological hypersurface* provided that for each $p \in S$, there is a neighborhood U of p in T and a homeomorphism of U onto an open set in $\mathbf{R}^n$ such that $\phi(U \cap S) = \phi(U) \cap \Pi$, where Π is a hyperplane in $\mathbf{R}^n$.

topological manifold An *n-dimensional topological manifold* T is a Housdorff space such that each point has a neighborhood homeomorphic to an open set in $\mathbf{R}^n$.

topological space A *topsp* is a set $\mathcal{S}$ together with a collection $\mathcal{O}$ of subsets called *open sets* such that

i. $\emptyset \in \mathcal{O}$ and $S \in \mathcal{O}$;

ii. If $U_1, U_2 \in \mathcal{O}$, then $U_1 \cap U_2 \in \mathcal{O}$;

iii. The union of any collection of open sets is open.

For such a topological space, the *closed sets* are the elements of the set

$$\{A|\mathcal{C}A \in \mathcal{O}\},$$

where $\mathcal{C}$ denote the *complement*, $\mathcal{C}A = S\ A = \{s \in S|s \notin A\}$. An *open neighborhood of a point* u in a topological space S is an open set U such that $u \in U$. Similarly, for a subset A of S, U is an *open neighborhood of* A if U is open and $A \subset U$.

topological sum Suppose E is a space that is the union of disjoint subspaces E_α, each of which is open (and closed) in E. Then E is called the *topl sum* of the spaces E_α and is denoted $E = \sum E_\alpha$.

topologist's sine curve The *topologist's sine curve* is the subspace of $\mathbf{R}^2$ consisting of all points $(x, \sin\frac{1}{x})$ for $0 \leq x < 1$, and all points $(0, y)$ for $-1 \leq y \leq 1$.

torsion coefficients *refer to* Fundamental theorem of finitely generated abelian groups

torsion-free connection For a connection D, D is called *torsion-free* if its torsion tensor is zero; that is, $[X, Y] = D_X Y + D_Y X$.
facta. For a semi-Riemannian manifold, the Levi-Civita connection is torsion-free.

torsion-free group A group is *torsion-free* if the subgroup generated by each element except the identity is infinite.

torsion tensor For an arbitrary connection D on a manifold M, the *torsion tensor of D* is a (1,2) tensor field on M satisfying

$$T(X, Y) = [X, Y] - D_X Y + D_Y X,$$

where $X, Y \in \mathfrak{X}(M)$.

torsion subgroup Let G be an abelian group. The set of all elements of finite order in G is a subgroup of G and is called the *torsion subgroup of G*.

totally geodesic A semi-Riemannian submanifold M of $\overline{M}$ is *totally geodesic* provided its shape tensor vanishes: $II = 0$.
facta. The followings are equivalent:

i. M is totally geodesic in $\overline{M}$.
ii. Every geodesic of M is also a geodesic of $\overline{M}$.
iii. If $v \in T_p\overline{M}$ is tangent to M, then the $\overline{M}$ geodesic γ_v lies initially in M.
iv. If α is a curve in M and $v \in T_{\alpha(0)}M$ then parallel translation of v along α is the same for M and for $\overline{M}$.

totally umbilic A semi-Riemannian submanifold M of $\overline{M}$ is *totally umbilic* provided every point of M is umbilic. Then there is a smooth normal vector field Z on M called the *normal curvature vector field* of M such that $II(V, W) = \langle V, W \rangle Z$ for all $V, W \in \mathfrak{X}(M)$.
facta. 1. A totally geodesic submanifold is a totally umbilic submanifold for which $Z = 0$.
2. The complete, connected, totally umbilic hypersurfaces of $\mathbf{R}^n_\nu$ $(n \geq 3)$ exactly the nondegenerate hyperplanes of $\mathbf{R}^n_\nu$ and the components of hyperquadrics .

totally vicious For a Lorentz manifold M, M is said to be *totally vicious* provided $I(p) \equiv I^+(p) \cap I^-(p) = M$ for all $p \in M$.

total manifold *refer to* vector bundle

trace of a homomorphism If G is a free abelian group with basis $e_1, \ldots, e_n$ and if $\phi : G \longrightarrow G$ is a homomorphism, the *trace of* ϕ is defined by the number $\mathsf{trace}A$, where A is the matrix of ϕ relative to the given basis. This number is independent of the choice of basis.
rel. trace of a matrix

trace of a matrix If $A = (a_{ij})$ is an $n \times n$ square matrix, then the *trace of* A, denoted $\mathsf{trace}A$, is defined by

$$\mathsf{trace}A = \sum_{i=1}^{n} a_{ii}.$$

If A and B are $n \times n$ matrices, then

$$\mathsf{trace}AB = \sum_{i,j} a_{ij} b_{ji} = \mathsf{trace}BA.$$

transitive law *refer to* equivalence relation

transvection An isometry $\phi : M \longrightarrow N$ is a *transvection along a geodesic* $\gamma : \mathbf{R} \longrightarrow M$ provided

i. ϕ translates γ; that is, $\phi(\gamma(s)) = \gamma(s+c)$ for all $s \in \mathbf{R}$ and some c.

ii. $\mathrm{d}\phi$ gives parallel translation along γ; that is, if $x \in T_{\gamma(s)}M$, then $\mathrm{d}\phi(x) \in T_{\gamma(s+c)}M$ is the parallel translate of x along γ.

transitive action *refer to* action of a Lie group

transversal *refer to* variation

triangle A simple region which has only three vertices with external angles $\alpha_i \neq 0$, $i = 1, 2, 3$, is called a *triangle*.

triangle inequality *refer to* metric

triangulation of a region A *triangulation of a regular region* $R \subset S$ is a finite family $\mathcal{T}$ of triangles T_i, $i = 1, \ldots, n$, such that

i. $\bigcup_{i=1}^{n} T_i = R$;

ii. If $T_i \cap T_j \neq 0$, then $T_i \cap T_j$ is either a common edge of T_i and T_j or a common vertex of T_i and T_j.

trivial bundle A k-vector bundle (E, π) over a manifold M is *trivial* provided it has k linearly independent global sections, or equivalently a global bundle chart $\phi : M \times \mathbf{R}^k \approx E$.
facta. $T\mathbf{R}^n$ and TS^1 are trivial.

trivial covering A covering $k : \widetilde{M} \longrightarrow M$ is *trivial* if each component of M is evenly covered by k.
facta. 1. If M is connected, k is diffeomorphism of each component C of $\widetilde{M}$ onto M, so $\lambda(k|C)^{-1}$ is a global cross section of k.
2. Every covering of simply connected manifold is trivial.

trivial top let S be a topological space. The topology in which $\mathcal{O} = \{\emptyset, S\}$ is called the *trivial topology*.

truncated Kruskal spacetime *refer to* Kruskal spacetime

U

umbilic A point p of $M \subset \overline{M}$ is *umbilic* provided there is a normal vector $z \in T_pM^\perp$ such that

$$II(v,w) = \langle v,w\rangle z$$

for all $v, w \in T_pM$. Then z is called the *normal curvature vector of M at p.*

underlying space of simplicial complex *refer to* polytope

uniform norm *refer to* L^2 norm

unit n-ball Let M be a semi-Riemannian manifold. The *unit n-ball B^n* is the set of all points p in M for which $|p| \leq 1$.

unit sphere Let M be a semi-Riemannian manifold. The *unit sphere S^{n-1}* is the set of all points p in M for which $|p| = 1$.

universal anti-de Sitter spacetime The four-dimensional nonflat Minkowski space $\widetilde{H}_1^4(r)$ is *universal anti-de Sitter spacetime* .

universal constant in Einstein field equation *refer to* Einstein field equation

upper hemisphere *refer to* hemisphere

V

vacuum If the stress-energy tensor T of a manifold M is zero, that is, if M is Ricci flat, then M is said to be *vacuum*.

variation A *variation of a curve segment* $\alpha : [a, b] \longrightarrow M$ is a two-parameter mapping

$$\mathbf{x} : [a, b] \times (-\delta, \delta) \longrightarrow M$$

such that $\alpha(u) = \mathbf{x}(u, 0)$ for all $a \leq u \leq b$. The u-parameter curves of a variation are called *longitudinal* and the v-parameter curves *transversal*. The *base curve* of $\mathbf{x}$ is α.

(first) variation formula Let $\alpha : [a, b] \longrightarrow M$ be a piecewise smooth curve segment with constant speed $c > 0$ and sign ε. If $\mathbf{x}$ is a variation of α, then

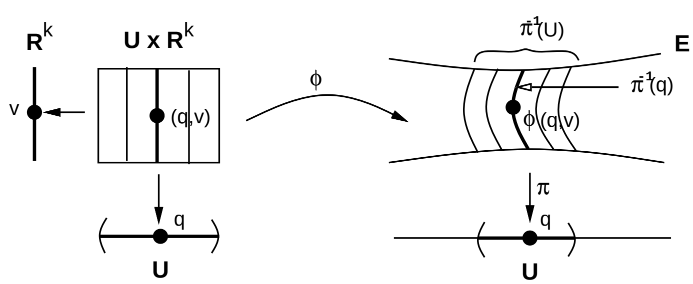
$$L'(0) = -\frac{\varepsilon}{c} \int_a^b \langle \alpha'', V \rangle \mathrm{d}u - \frac{\varepsilon}{c} \sum_{i=1}^{k} \langle \triangle \alpha'(u_i), V(u_i) \rangle + \frac{\varepsilon}{c} \langle \alpha', V \rangle \Big|_a^b,$$

where $u_1 < \cdots < u_k$ are the breaks of α and $\mathbf{x}$.
cf. Synge's formula for second variation

vector bundle A *k-vector bundle (E, π) over a manifold M* consists of a manifold E and a smooth map $\pi : E \longrightarrow M$ such that

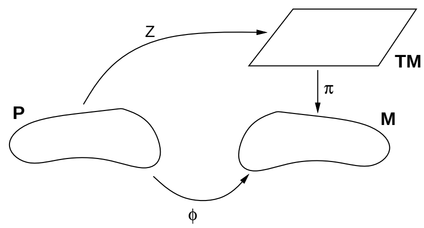
i. each $\pi^{-1}(p)$, $p \in M$ is a k-dimensional vector space ,
ii. for each $p \in M$, there is a neighborhood U of p in M and a diffeomorphism $\phi : U \times \mathbf{R}^k \longrightarrow \pi^{-1}(U) \subset E$ such that for each $q \in U$, the map $v \rightarrow \phi(q, v)$ is a linear isomorphism from $\mathbf{R}^k$ onto $\pi^{-1}(q)$.

From above, M is called the *base manifold*, E the *total manifold*, π the *projection*, $\pi^{-1}(p)$ the *fiber over* p, $\mathbf{R}^k$ the *standard fiber*, and ϕ a *bundle chart* of the given bundle.



vector field A *vector field V on a manifold M* is a function that assigns a tangent vector V_p to M at p to each point $p \in M$.

A *vector field Z on a smooth map $\phi : P \longrightarrow M$* is a mapping $Z : P \longrightarrow TM$ such that $\pi \circ Z = \phi$, where π is the projection $TM \rightarrow M$.



velocity vector The *velocity vector* of a curve $\alpha : I \longrightarrow M$ at $t \in I$ is

$$\alpha'(t) = \mathrm{d}\alpha \left(\left. \frac{\mathrm{d}}{\mathrm{d}u} \right|_t \right) \in T_{\alpha(t)}M.$$

vertex scheme If K is a simplicial complex, let V be the vertex set of K. Let $\mathcal{K}$ be the collection of all subsets $\{a_0, \ldots, a_n\}$ of V such that the vertices $a_0, \ldots, a_n$ span a simplex of K. The collection $\mathcal{K}$ is called the *vertex scheme of K*.

vertex of n-simplex *refer to* 1. n-simplex
2. skeleton

vertex set *refer to* abstract simplicial complex

viral function *refer to* momentum function

viral theorem Let M be a semi-Riemannian manifold, K the kinetic energy function and $V : M \longrightarrow \mathbf{R}$ a given potential. Let

$$L(v) = K(v) - V(\tau_M v)$$

be usual Lagrangian and $\tau_M : TM \longrightarrow M$ the canonical projection. Let X be a vector field on M and e a regular value of E. Assume the level set Σ_e at e is compact. The the time and space averages of $G(X)$ on Σ_e are both zero.

volume element A *volume element* on an n-dimensional semi-Riemannian manifold M is a smooth n-form ω such that $\omega(e_1, \ldots, e_n) = \pm 1$ for every frame on M.

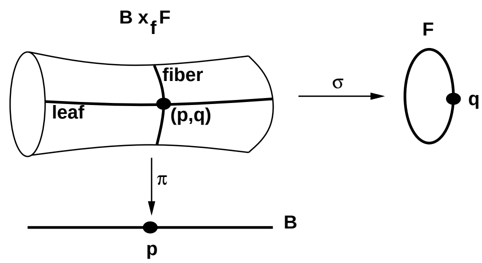
facta. 1. Volume elements always exist at least locally.
2. A semi-Riemannian manifold M has a (global) volume element if and only if M is orientable .

W

warped product Let B and F be semi-Riemannian manifolds and let $f > 0$ be a smooth function on B. The *warped product* $M = B \times_f F$ is the product manifold $B \times F$ furnished with metric tensor

$$\mathbf{g} = \pi^\star(\mathbf{g}_B) + (f \circ \pi)^2 \sigma^\star(\mathbf{g}_F),$$

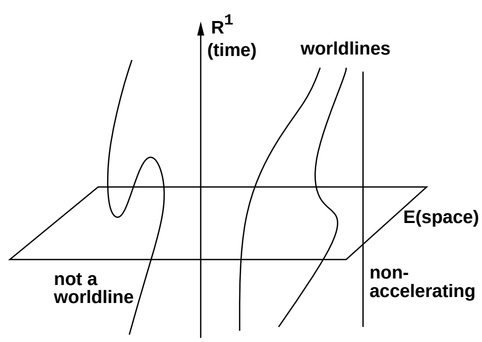
where π and σ are the projections of $B \times F$, respectively. The function f is called *warping function* .



warping function *refer to* warped product

wedge product *refer to* exterior algebra

worldline A *worldline* in Newtonian spacetime is a one-dimensional submanifold W such that $T|_W$ is a diffeomorphism onto an interval $I \subset \mathbf{R}^1$.



Z

zig-zag lemma Suppose one is given chain complexes $\mathcal{C} = \{C_p, \partial_C\}$, $\mathcal{D} = \{D_p, \partial_D\}$ and $\mathcal{E} = \{E, \partial_E\}$, and chain maps ϕ, ψ such that the sequence

$$0 \longrightarrow \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0$$

is exact. Then there is a long exact homology sequence

$$\cdots \longrightarrow H_p(\mathcal{C}) \xrightarrow{\phi_\star} H_p(\mathcal{D}) \xrightarrow{\psi_\star} H_p(\mathcal{E}) \xrightarrow{\partial_\star} H_{p-1}(\mathcal{C}) \xrightarrow{\phi_\star} H_{p-1}(\mathcal{D}) \longrightarrow \cdots,$$

where $\partial_\star$ is induced by the boundary operator in $\mathcal{D}$.

Terminologies in Differential Geometry

with special focus on differential topology, semi-Riemannian manifolds, relativity and gravitation

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There are many ways to represent a phenomenon in various ways. Authors of each book would like to find the best way to represent their ideas to convey them as clear as possible. As an unexpected result, we often face different types of notations and definitions describing the same concept. It makes us difficult to grasp the concept of the terms.

Differential geometry is not an exception, especially, semi-Riemannian geometry due to its inter-disciplinary nature. I believe that it would be helpful to have a collection of terminologies described in a uniform idea and notions. I collected and re-wrote terminologies used in differential geometry, especially in differential topology and semi-Riemannian manifolds. Some terms used in relativity and gravitation were added to this book and represented in mathematical viewpoints rather than in physical viewpoints, since physical representations often become a barrier for mathematicians to understand the underlying mathematical meaning.